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PERTURBATION OF INITIAL-VALUE EQUATIONS IN GENERAL RELATIVITY

by

CHING-WO NG ©

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled PERTURBATION OF INITIAL-VALUE EQUATIONS IN GENERAL RELATIVITY submitted by Ching-wo Ng in partial fulfillment of the requirements for the degree of Master of Science.

ABSTRACT

The main theme of this thesis is to investigate the property of a single minimal surface under small perturbation. The method consists mainly of finding the area of the perturbed minimal surface in a momentarily static initial hypersurface. According to the Israel-Carter conjecture, the perturbed minimal surface will eventually settle down to a Schwarzschild (or Reissner-Nordstrom) or Kerr event horizon. (In stationary spacetime, the event horizon is necessarily the minimal surface.) The perturbed minimal surface and the final, stationary event horizon are compared and studied. The result then shows that the Penrose conjecture of cosmic-censorship cannot be violated through this type of perturbation.

Both the Schwarzschild and Kerr initial data are studied under the above scheme. However, the Ernst formulation for an axisymmetric, (t,ϕ) reversible space-time is found to be unsuitable for this type of investigation. A new formulation is set up and studied extensively. It is found that some advantages of Ernst formulation are lost in the new formulation.

A brief outline of this thesis is listed below: Chapter 1 reviews the Cauchy problem in general relativity. Equations necessary for later chapters are derived.

No new material resides in this chapter. In Chapter 2, the time-symmetric initial value problem is formulated for calculation in next chapter. Except for some small innovations, it is essentially the same as Brill's formulation of time-symmetric gravitational waves. Chapter 3 contains the perturbation of the Schwarzschild and Reissner-Nordstrom initial data. The area of the minimal surface is calculated and is shown to be in accordance with Penrose's conjecture. Chapter 4 contains the new formulation and the perturbation of the Kerr initial data. In Chapter 5, the Israel-Leibovitz theorem is generalized to show that the Penrose conjecture holds for a charged spherical collapse.

In the last three chapters, except section II of Chapter 3, all materials are the author's own work.

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NOTATIONS

In this thesis, the following notations are used:

(1) Latin indices range from 1 to 3,

i.e. $i,j,k,\ldots = 1,2,3$.

(2) Greek indices range from 1 to 4,

i.e. $\alpha,\beta,\gamma,\ldots = 1,2,3,4$.

(3) Signature : $\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$

(4) Commas "," stand for partial differentiation,

i.e. $A_{\mu,\nu} \equiv \partial A_{\mu}/\partial x^{\nu}$.

(5) Strokes "|" and "∇" stand for covariant differentiation.

(6) The Reimannian tensor is defined by

$$R^{\alpha}{}_{\beta\gamma\delta} \equiv \Gamma^{\alpha}_{\beta\delta,\gamma} - \Gamma^{\alpha}_{\beta\gamma,\delta} + \Gamma^{\alpha}_{\mu\delta}\Gamma^{\mu}_{\beta\gamma} - \Gamma^{\alpha}_{\mu\gamma}\Gamma^{\mu}_{\beta\delta}.$$

CHAPTER 1

CAUCHY'S PROBLEM IN GENERAL RELATIVITY

I. Introduction

A brief description of the Cauchy's problem in general relativity is given in this chapter, following more or less the historical development. Emphasies are put on equations that are needed in the other chapters. The source of this chapter is mainly from the texts and some of the papers listed in the bibliography.

The Kerr solution and Schwarzschild solution represent fields of idealized physical system. For more general case, one has to recourse to series solution in powers of x^4 or other numerical methods. Such a method is closely related to the initial-value problem, also known as the Cauchy problem. Besides its practical value, initial-value problem also brings insights and understandings into the structure of the Einstein field equation and the spacetime manifold.

II. "Classical" Treatment

Consider a hypersurface Σ, labelled $x^4 = t$ imbedded in the Riemannian spacetime manifold V.

Suppose $g_{\mu\nu}$ and $g_{\mu\nu,4}$ are given on Σ. A casual glance at the Einstein tensor $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ gives

the impression that all ten second time-derivative of $g_{\mu\nu}$ can be expressed in terms of the initial data $g_{\mu\nu}$ and $g_{\mu\nu,4}$ on Σ. This, however, is deceptive. The Einstein tensor $G_{\mu\nu}$ satisfies the Bianchi identity,

$$G^{\mu\nu}{}_{|\mu} = 0 \ . \tag{1}$$

More explicitly,

$$G^{4\nu}{}_{,4} = -G^{i\nu}{}_{,i} - \Gamma^{\mu}_{\mu\lambda}G^{\lambda\nu} - \Gamma^{\nu}_{\lambda\mu}G^{\mu\lambda} \ . \tag{2}$$

The highest time-derivative in the right hand side is of second order. This then clearly shows that $G^{4\nu}$ cannot contain second-order time derivative. In other words, the Einstein equation

$$G_{4\mu} = -\kappa T_{4\mu} \tag{3}$$

contains no information about the time evolution of the initial data. Instead, it represents a set of constraints to which the initial data must satisfy.

The Bianchi identity of (1) combined with the Einstein field equation gives

$$T^{\mu\nu}{}_{|\mu} = 0 \ . \tag{4}$$

Following (2), this can be expressed as

$$T^{4\nu}{}_{,4} = -T^{i\nu}{}_{,i} - \Gamma^{\mu}_{\lambda\mu}T^{\lambda\nu} - \Gamma^{\nu}_{\lambda\mu}T^{\mu\lambda} \ . \tag{5}$$

The interpretation of this is that $T^{\mu\nu}$ cannot be

prescribed freely throughout the spacetime; instead, equation (4) must be followed.

As four of the ten Einstein equations are used as constraints, only six of ten $g_{\mu\nu}$ can be determined by the rest of the Einstein equations. The remaining four $g_{\mu 4}$ must be set arbitrarily. This arbitrariness is an unremovable feature of general relativity reflecting the principle of covariance, namely that any equation of physical law must preserve its form under general coordinate transformation.

To sum up, precribe four arbitrary functions for $g_{\mu 4}$ (say, 0,0,0,-1) which in turn fixes the coordinates, then prescribe g_{ij} and $g_{ij,4}$ on $\sum$ in such a way that $G_{\mu 4} = -\kappa T_{\mu 4}$ holds. One can then proceed and solve the remaining field equations

$$G_{ij} = -\kappa T_{ij} \quad .$$

As for the prescription of $T_{\mu\nu}$, besides constraints (3) $T_{\mu\nu}$ also has to satisfy (4).

III. The "Variational Principle" Approach

Through the variational principle, one can recognize what quantity must be held fixed at the limits. This in turn gives one the idea of what initial data to prescribe.

In the Hilbert (1915) - Palatini (1919) formulation, the Lagrangian density is taken to be

$$\mathcal{R} \equiv \sqrt{-g}\, R \tag{6}$$

where R is the Riemannian scalar, and the quantities to be varied are the ten components of the metric tensor, $g_{\mu\nu}$, and the forty components of the affine connection, $\Gamma^{\alpha}_{\beta\gamma}$. These quantities are held fixed on the boundary of the spacetime manifold.

In the Arnowitt-Deser-Misner formulation, the spacetime is split into a spacelike hypersurface $\sum$ and a timelike direction normal to $\sum$. The Lagrangian density of (6) is expressed in terms of quantities intrinsic to $\sum$, namely the 3-Riemannian scalar, ${}^3R$, and the extrinsic curvature K_{ij}, i.e.

$$\begin{aligned} {}^4R &= {}^4R^{ij}{}_{ji} + 2\,{}^4R^{4i}{}_{i4} \\ &= {}^3R + (\underline{n}\cdot\underline{n})[(\mathrm{Tr}\ K)^2 - \mathrm{Tr}(K^2)] \\ &\quad + \text{covariance divergence} . \end{aligned} \tag{7}$$

The variational principle for vacuum field in ADM's language now has the form

$$\delta\int\{{}^3R + (\underline{n}\cdot\underline{n})[(\mathrm{Tr}\ K)^2 - \mathrm{Tr}(K^2)]\}N\sqrt{{}^3g}\ dx_1 dx_2 dx_3 dt = 0 \tag{8}$$

where $\underline{n}$ is a unit vector normal to $\sum$, and

$$\underline{n}\cdot\underline{n} = \begin{cases} 1 & \text{if} \quad \underline{n} \text{ is spacelike} \\ -1 & \text{if} \quad \underline{n} \text{ is timelike} \end{cases} \quad ;$$

and N is the lapse function measuring the proper time between the two spacelike hypersurfaces, and

$$\mathrm{Tr}\, K \equiv K^i{}_i \qquad \text{and} \qquad \mathrm{Tr}(K^2) \equiv K^i{}_j K^j{}_i \quad .$$

At this moment, we digress to investigate the ADM's 3+1 split of spacetime. To construct a spacetime manifold from a stack of spacelike hypersurfaces Σ, one needs to know (1) the 3-geometry of Σ, g_{ij}, (2) the lapse function N measuring the proper time between two spacelike hypersurfaces, and (3) the shift function N^i measuring the spatial shift of two corresponding points on two adjacent hypersurfaces. In terms of these functions, the 4-line-element

$$^{(4)}ds^2 = g_{ij}(dx^i + N^i dt)(dx^j + N^j dt) - N^2 dt. \qquad (9)$$

Comparing with $^4ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$, one arrives at

$$\begin{pmatrix} {}^{(3)}g_{ij} & N_i \\ N_j & N_i N^i - N^2 \end{pmatrix} = \begin{pmatrix} g_{ij} & g_{i4} \\ g_{4j} & g_{44} \end{pmatrix} \quad , \qquad (10)$$

or equivalently

$$\begin{pmatrix} {}^3g^{ij} & \frac{N^i}{N^2} \\ \frac{N^j}{N^2} & -\frac{1}{N^2} \end{pmatrix} = \begin{pmatrix} g^{ij} - \frac{N^iN^j}{N^2} & g^{i4} \\ g^{4j} & g^{44} \end{pmatrix} . \tag{11}$$

The proper volume $\sqrt{-{}^{(4)}g}\, dx^1dx^2dx^3dx^4$ acquires the form

$$N\sqrt{g}\, dx^1dx^2dx^3dt \quad . \tag{12}$$

A spacelike hypersurface Σ is uniquely specified by a pair of symmetric 3-tensor $g_{ij}(x^k)$ and $K_{ij}(x^k)$ which determine the intrinsic geometry of Σ and the manner Σ is imbedded in the enveloping spacetime respectively. The extrinsic curvature K_{ij} is defined by (Israel, 1966)

$$\frac{\delta n^\alpha}{\delta \xi^i} = - K_i{}^j\, e^\alpha_{(j)} \tag{13}$$

where $\delta/\delta\xi^i$ means absolute derivative with respect to ξ^i, the intrinsic coordinates of Σ. In covariant form

$$K_{ij} = -e^\alpha_{(j)}\, \frac{\delta n_\alpha}{\delta \xi^i} = -e^\alpha_{(j)} \left(\frac{\partial n_\alpha}{\partial \xi^i} - n_\beta \Gamma^\beta_{\alpha\gamma}\, \frac{\partial x^\gamma}{\partial \xi^i}\right) . \tag{14}$$

By taking Σ to be x^4 = constant and choosing $x^i = \xi^i$, we have

$$e^\alpha_{(j)} \equiv \frac{\partial x^\alpha}{\partial \xi^j} = \delta^\alpha_j \quad ; \quad \text{and} \quad \frac{\partial x^\gamma}{\partial \xi^i} = \delta^\gamma_i \quad .$$

The unit vector normal to Σ, n_α, is given by

n_α = (normalization factor) $\times\, \partial_\alpha x^4 = -N\delta_\alpha^4$.

With these, (14) becomes

$$K_{ij} = -N\,\Gamma^4_{ij}$$

$$= -\frac{N}{2}\{g^{44}(g_{4i,j} + g_{4j,i} - g_{ij,4}) + g^{4k}(g_{ki,j} + g_{kj,i} - g_{ij,k})\}.$$

With the help of (11), this becomes

$$K_{ij} = \frac{1}{2N}\{N_{i,j} + N_{j,i} - g_{ij,4} + 2N_k\Gamma^k_{ij}\}$$

$$= \frac{1}{2N}\{N_{i|j} + N_{j|i} - g_{ij,4}\} \quad . \tag{15}$$

In terms of g_{ij} and K_{ij}, the constraint equations $G_\mu{}^4 = -\kappa T_\mu{}^4$ are

$$\tfrac{1}{2}\,{}^3R + \frac{1}{\underline{n}\cdot\underline{n}}\,[(\mathrm{Tr}\ K)^2 - (\mathrm{Tr}\ K)^2] = \kappa T_4{}^4 \tag{16}$$

and

$$\frac{1}{\underline{n}\cdot\underline{n}}\,[K_j{}^i{}_{|i} - K_i{}^i{}_{|j}] = -\kappa T_j{}^4 \tag{17}$$

where the stroke "|" stands for covariant differentiation with respect to g_{ij}.

Return to variational principle. By introducing the super-Hamiltonian $\mathcal{H}$, the super-momentum $\mathcal{H}^i$ and field momentum π_{ij}, equation (8) can be expressed as

$$\delta\int[\pi^{ij}g_{ij,4} - N\mathcal{H} - N_i\mathcal{H}^i]d^4x = 0 . \tag{18}$$

Explicitly,

$$\mathcal{H} \equiv g^{-\frac{1}{2}}[\mathrm{Tr}(\pi^2) - \tfrac{1}{2}(\mathrm{Tr}\ \pi)^2] - g^{\frac{1}{2}}\,{}^3R \tag{19}$$

$$\mathcal{H}^i \equiv -\ 2\pi^{ij}{}_{|j} \tag{20}$$

and

$$\pi^{ij} \equiv \sqrt{g}[g^{ij}(\mathrm{Tr}\ K) - K^{ij}] . \tag{21}$$

In this formulation, the quantities to be varied freely, but held fixed on the initial and final spacelike hypersurfaces, are π^{ij}, g_{ij}, N, and N^i. The first two play the role of initial (and final) data on $\sum$ and the latter two play the role of Langrange multiplier giving the constraints

$$\left.\begin{aligned} \mathcal{H} &= 0 \\ \text{and}\quad \mathcal{H}^i &= 0 \end{aligned}\right\} \tag{22}$$

Besides providing information about the initial data, the main attraction of this formulation comes from its application in the quantization of gravitational field. For a lucid explanation of this method, see Kuchar (1972).

To sum up, in a vacuum spacetime, the intrinsic metric g_{ij} and the extrinsic curvature K_{ij} of a hypersurface $\sum$ (t = const) determine uniquely a solution of

the Einstein vacuum field equation if and only if g_{ij} and K_{ij} satisfy the equations (Israel, 1975),

$$\tfrac{1}{2}\,{}^{3}R + \mathrm{Tr}(K^{2}) - (\mathrm{Tr}\ K)^{2} = 0$$

and

$$K_{i}{}^{j}{}_{|j} - (\mathrm{Tr}\ K)_{|i} = 0\ .$$

CHAPTER 2

TIME-SYMMETRIC IVP

I. Introduction

In this and subsequent chapters, we turn our attention from the general IVP to some specific ones. The ones we are interested in are those with time-symmetric initial data and those with what Hawking (1973) termed (t,ϕ) reversible initial data, both on a conformally flat spacelike hypersurface. The IVP with such symmetries are reformulated to allow detailed calculation. Some classes of perturbations on the initial data are also calculated in the fashion proposed by Penrose in his attempt to construct a naked singularity. The essential idea was to construct time-symmetric initial data which has smaller mass than the Schwarzschild solution but with the same, or larger area of apparent horizon. Such a case obviously would contradict Hawking's "second law of Black Hole dynamics" which states that in a strongly asymptotically predictable spacetime free of "naked singularities", the area of a black hole must always increase with time.

In Chapter 2, a brief review of the idea of symmetry with its bearing on metric tensor is presented in Section II. In the rest of the chapter, the time-symmetric IVP is defined and formulated. Except for

some innovations, the essence of the formulation comes from Brill's (1959) investigation on time-symmetric gravitational waves.

In Chapter 3, the perturbation is carried out on a time-symmetric, conformally flat spacelike hypersurface $\sum_i$ with Schwarzschild initial data. The Reissner-Nordstrom metric is also discussed. Except for Section II, in which information necessary for the rest of the chapter is explained, the main body of this chapter contains original work.

In Chapter 4, initial data with both axisymmetry and (t,ϕ) reversal is studied. A new formulation for axisymmetric gravitational field is presented. The Kerr solution in this formulation is studied. The perturbation of Kerr solution is also performed. All informations, unless otherwise specified, are original.

In Chapter 5, the Israel-Leibovitz Theorem is generalized to charged spherical collapse. The result is then used to show that Penrose cosmic censorship conjecture holds in such a collapse. This chapter, like the previous one, is original work.

II. A Brief Review on Symmetry

From the palaeolithic cave paintings to the ultramodern pop-arts, symmetry has always been the unfailing carrier for beauty, harmony and perfection. Though it is universally appreciated, symmetry, as beauty, is in the eyes of the beholder. Its understanding, hence its definition, varies from crude to refined according to the power of perception of the beholder. The most general, and mathematically applicable, definition is, perhaps, given by Hermann Weyl in his book "Symmetry". Weyl defines symmetry to be the property that enables a system to remain unchanged after certain operation. In such a case, the system is said to be symmetric with respect to such an operation.

Spacetime manifold, on most occasions, admits some groups of approximate symmetry transformations. Such symmetries can be used to deduce information about the metric tensor. We will show how this is done in this section.

Consider the spacetime manifold V with coordinate x^μ and metric tensor $g_{\mu\nu}(x^\alpha)$.

Under Weyl's definition, the spacetime V is said to possess a hidden symmetry if there exists a coordinate transformation

$$x^{\mu} \to \bar{x}^{\mu} \tag{1}$$

such that the transformed metric $\bar{g}_{\mu\nu}(\bar{x}^{\alpha})$ is the <u>same function</u> of its argument $\bar{x}^{\alpha}$ as the original metric $g_{\mu\nu}(x^{\alpha})$ is of its argument x^{α}, that is, $g_{\mu\nu}$ is invariant <u>in form</u> under such a transformation.

The two metrics are related by the tensor transformation formula,

$$g_{\mu\nu}(x^{\gamma}) = \bar{g}_{\alpha\beta}(\bar{x}^{\gamma}) \frac{\partial \bar{x}^{\alpha}}{\partial x^{\mu}} \frac{\partial \bar{x}^{\beta}}{\partial x^{\nu}} \quad . \tag{2}$$

If $g_{\mu\nu}$ is invariant in form under such a transformation, one can replace $\bar{g}_{\mu\nu}(\bar{x}^{\gamma})$ by $g_{\mu\nu}(\bar{x}^{\gamma})$. This gives

$$g_{\mu\nu}(x^{\gamma}) = g_{\alpha\beta}(\bar{x}^{\gamma}) \frac{\partial \bar{x}^{\alpha}}{\partial x^{\mu}} \frac{\partial \bar{x}^{\beta}}{\partial x^{\nu}} \quad . \tag{3}$$

Coordinate transformation that satisfies equation (3) is called an isometry. In a more precise mathematical language, an isometry is a diffeomorphism $\phi: V \to V$ that carries the metric tensor g into itself, that is, the mapped metric $\phi(g)$ is equal to g everywhere.

To expedite physical understanding, we leave the general case for a while and treat the special case of the infinitesimal transformation

$$x^{\mu} \to \bar{x}^{\mu} = x^{\mu} + \varepsilon\xi^{\mu} \quad . \tag{4}$$

For such a transformation,

$$\frac{\partial \bar{x}^{\alpha}}{\partial x^{\mu}} = \delta^{\alpha}_{\mu} + \varepsilon \xi^{\alpha}{}_{,\mu} \tag{5}$$

hence

$$\begin{aligned}(\frac{\partial \bar{x}^{\alpha}}{\partial x^{\mu}})(\frac{\partial \bar{x}^{\beta}}{\partial x^{\nu}}) &= (\delta^{\alpha}_{\mu} + \varepsilon \xi^{\alpha}{}_{,\mu})(\delta^{\beta}_{\nu} + \varepsilon \xi^{\beta}{}_{,\nu}) \\ &= \delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} + \varepsilon(\delta^{\alpha}_{\mu}\xi^{\beta}{}_{,\nu} + \delta^{\beta}_{\nu}\xi^{\alpha}{}_{,\mu}) + 0(\varepsilon^2) \qquad (6)\end{aligned}$$

and, from Taylor's expansion,

$$\begin{aligned}g_{\alpha\beta}(\bar{x}^{\gamma}) &= g_{\alpha\beta}(x^{\gamma} + \varepsilon\xi^{\gamma}) \\ &= g_{\alpha\beta}(x^{\gamma}) + \varepsilon\xi^{\gamma}\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} + 0(\varepsilon^2) \; . \qquad (7)\end{aligned}$$

With the help of (6) and (7), equation (3) reduces to, up to the order of ε,

$$g_{\mu\beta}\xi^{\beta}{}_{,\nu} + g_{\alpha\nu}\xi^{\alpha}{}_{,\mu} + \xi^{\alpha}g_{\mu\nu,\alpha} = 0 \; . \tag{8}$$

The covariant component ξ_{α} is defined by

$$\xi_{\alpha} \equiv g_{\alpha\beta}\xi^{\beta} \; . \tag{9}$$

A simple differentiation gives

$$\xi_{\mu,\nu} = (g_{\mu\beta}\xi^{\beta})_{,\nu} = g_{\mu\beta,\nu}\xi^{\beta} + g_{\mu\beta}\xi^{\beta}{}_{,\nu} \; . \tag{10}$$

Similarly,

$$\xi_{\nu,\mu} = g_{\nu\beta,\mu}\xi^{\beta} + g_{\nu\beta}\xi^{\beta}{}_{,\mu} \; . \tag{11}$$

Equation (8) can thus be written in covariant form

$$\xi_{\mu,\nu} + \xi_{\nu,\mu} - (g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha})\xi^{\alpha} = 0 . \qquad (12)$$

The bracket in equation (12) is twice the affine connection $\Gamma_{\mu\nu,\alpha}$, hence we have

$$\xi_{\mu,\nu} + \xi_{\nu,\mu} - 2\xi_{\alpha}\Gamma^{\alpha}_{\mu\nu} = 0 \qquad (13)$$

where

$$\Gamma^{\alpha}_{\mu\nu} \equiv g^{\alpha\beta}\Gamma_{\mu\nu,\beta} \quad .$$

In covariant derivative form, (13) becomes

$$\xi_{\mu|\nu} + \xi_{\nu|\mu} = 0 . \qquad (14)$$

This is the well-known Killing's equation, and ξ_{μ} is called the Killing vector. It is not hard to show (Hawking, 1972) that equation (14) is the necessary and sufficient condition for an isometry, hence the hidden symmetry of the spacetime, to exist. In other words, if the spacetime manifold admits a vector field ξ_{μ} that satisfies the Killing's equation, a symmetry exists.

To remove the doubt which arises from the use of the special transformation (4), a more general treatment is given below.

Consider the same spacetime V, coordinate x^α and metric $g_{\mu\nu}(x^\alpha)$.

Any equation

$$\bar{x}^\mu = f^\mu(x^\alpha, a) \tag{15}$$

where a is a parameter, defines a coordinate transformation for each value of a.

The set S of all such transformations forms an one-parameter continuous group of transformation G if the following group properties are met:

(1) Two such transformations f^μ, g^μ can be combined to give a third transformation of the form of eq. (15), i.e. if f^μ, g^μ belong to the set S then $h^\mu \equiv f^\mu[g^\nu(\bar{x}^\alpha), a]$ also belongs to S.

(2) There exists an identity transformation

$$x^\mu = f^\mu(x^\alpha, a_o)$$

where a_o is the value of the parameter for such transformation.

(3) For every transformation in the set S, a unique inverse transformation exists and belongs to the set S.

Such a transformation (15) satisfies a system of differential equations (Eisenhart, 1966)

$$\frac{d\bar{x}^\mu}{da} = \psi(a)\,\xi^\mu(\bar{x}^\alpha) \; . \tag{16}$$

Define a new parameter t by

$$t \equiv \int_{a_o}^{a} \psi(a)\, da$$

where a_o is the value of a for the identity transformation.

In terms of the new parameter t, eq. (16) reduces to

$$\frac{d\bar{x}^{\mu}}{dt} = \xi^{\mu}(\bar{x}^{\alpha}) \quad . \tag{17}$$

The identity transformation now has a parameter t = 0.

Consider $\bar{x}^{\mu}$ as a continuous function of t, and if $\xi^{\mu}(\bar{x}^{\alpha})$ is a well-behaved function in a neighbourhood of $t = 0$, $\bar{x}^{\mu}$ can be expressed in the form

$$\bar{x}^{\mu} = \bar{x}^{\mu}\Big|_{t=0} + \frac{d\bar{x}^{\mu}}{dt}\Big|_{t=0} t + \frac{d^2\bar{x}^2}{dt^2}\Big|_{t=0} \frac{t^2}{2!} + \ldots \tag{18}$$

but $\bar{x}^{\mu}$ evaluated at $t = 0$ is just the identity transformation, that is,

$$\bar{x}^{\mu}\Big|_{t=0} = x^{\mu} \;;$$

for the similar reason, and using equation (17), we have

$$\frac{d\bar{x}^{\mu}}{dt}\Big|_{t=0} = \xi^{\mu}(\bar{x}^{\alpha})\Big|_{t=0} = \xi^{\mu}(x^{\alpha}) \quad .$$

Also

$$\left.\frac{d^2\bar{x}^\mu}{dt^2}\right|_{t=0} = \xi^\beta(\bar{x}^\alpha) \left.\frac{\partial \xi^\mu(\bar{x}^\alpha)}{\partial \bar{x}^\beta}\right|_{t=0}$$

$$= \xi^\beta(x^\alpha) \frac{\partial \xi^\mu(x^\alpha)}{\partial x^\beta} \quad .$$

With these results, equation (18) reduces to

$$\bar{x}^\mu = x^\mu + \xi^\mu(x^\alpha)t + \xi^\beta(x^\alpha) \frac{\partial \xi^\mu(x^\alpha)}{\partial x^\beta} \frac{t^2}{2!} + \ldots \tag{19}$$

An infinitesimal transformation corresponds to a change in the parameter t from $t_1 = 0$ to $t_2 = \delta t$. Hence, from equation (19), an infinitesimal transformation has the form,

$$\bar{x}^\mu = x^\mu + \xi^\mu(x^\alpha)\delta t + \xi^\beta(x^\alpha) \frac{\partial \xi^\mu(x^\alpha)}{\partial x^\beta} \frac{(\delta t)^2}{2!} + \ldots$$

Neglecting $(\delta t)^2$ and higher order terms, this gives

$$\bar{x}^\mu = x^\mu + \xi^\mu(x^\alpha)\ dt \quad . \tag{20}$$

Following Eisenhart's notation, we introduce

$$Xf = \xi^\mu f_{,\mu} \quad . \tag{21}$$

[This is also known as the Lie derivative of a scalar function f along a vector field ξ^μ, i.e. $\mathcal{L}_\xi(f) \equiv \xi^\mu f_{,\mu}$]. This quantity Xf is called the generator of the one parameter transformation group given by equation (19) as it uniquely determines the infinitesimal transforma-

tion of equation (20), and the finite transformation of (19) is just repeated infinitesimal transformations.

Since we are interested in spacetime manifold with symmetries, consider those infinitesimal transformations that leave the metric tensor form-invariant, that is the isometric infinitesimal transformation. Suppose such a transformation is given by

$$\bar{x}^\mu = x^\mu + \xi^\mu \delta t ,$$

the condition that $g_{\mu\nu}$ is form-invariant requires that

$$g_{\mu\alpha}\xi^\alpha{}_{,\nu} + g_{\nu\alpha}\xi^\alpha{}_{,\mu} + \xi^\alpha g_{\mu\nu,\alpha} = 0 \quad . \tag{22}$$

After some juggling, this gives back the Killing's equation

$$\xi_{\mu|\nu} + \xi_{\nu|\mu} = 0 .$$

Arguing in reverse, we can see, at least crudely, the reason why a Killing vector field in the spacetime manifold implies some form of symmetry. If a spacetime manifold admits a Killing vector field ξ_μ, one can use ξ_μ to form a generator

$$Xf \equiv \xi^\mu f_{,\mu}$$

which generates a one-parameter continuous transformation group. This generator also uniquely determines an isometric infinitesimal transformation,

$$\bar{x}^{\mu} = x^{\mu} + \xi^{\mu}\,\delta t . \tag{23}$$

This transformation will then leave the metric form-invariant.

The Killing's equation, being in covariant form, holds in all coordinate system. Thus we can choose a very special coordinate in which the Killing vector ξ^{μ} has the form

$$\xi^{\alpha} = \delta^{\alpha}{}_{\kappa} . \tag{24}$$

That is, we choose the coordinate in such a way that one of the basis vectors, say $\underline{e}_{\kappa}$, coincide with the Killing vector $\underline{\xi}$.

In this particular coordinate, equation (22) reduces to

$$g_{\mu\nu,\alpha}\,\delta^{\alpha}{}_{\kappa} = 0 \tag{25}$$

since

$$\delta^{\alpha}{}_{\kappa,\nu} = \delta^{\alpha}{}_{\kappa,\mu} = 0 . \tag{26}$$

Summing over α in equation (25) gives

$$g_{\mu\nu,\kappa} = 0 . \tag{27}$$

In other words, the components of the metric tensor, $g_{\mu\nu}$, is independent of the κ^{th} coordinate! This important result fulfills our earlier claim that symmetry of the spacetime can be used to deduce information about

the metric tensor without solving the Einstein equation.

III. Time-Symmetric IVP

The time-symmetric IVP has been investigated in conjunction with various other problems by different authors, notably Weber and Wheeler (1957), Brill (1959), and Gibbons (1972).

By time-symmetry, we mean that the spacetime admits a spacelike hypersurface $\sum_i$ and an isometry that leaves every point on $\sum_i$ unchanged but reverses the direction of time for all other points. Consider a point P on a future-pointing timelike geodesic orthogonal to $\sum_i$. Suppose the proper distance from P to $\sum_i$ along the geodesic is $\Delta\tau$. The isometry will map P into P' on a past-pointing timelike geodesic orthogonal to $\sum_i$, with a proper distance $\Delta\tau$ measured in the reversed direction (see figure 2.1).

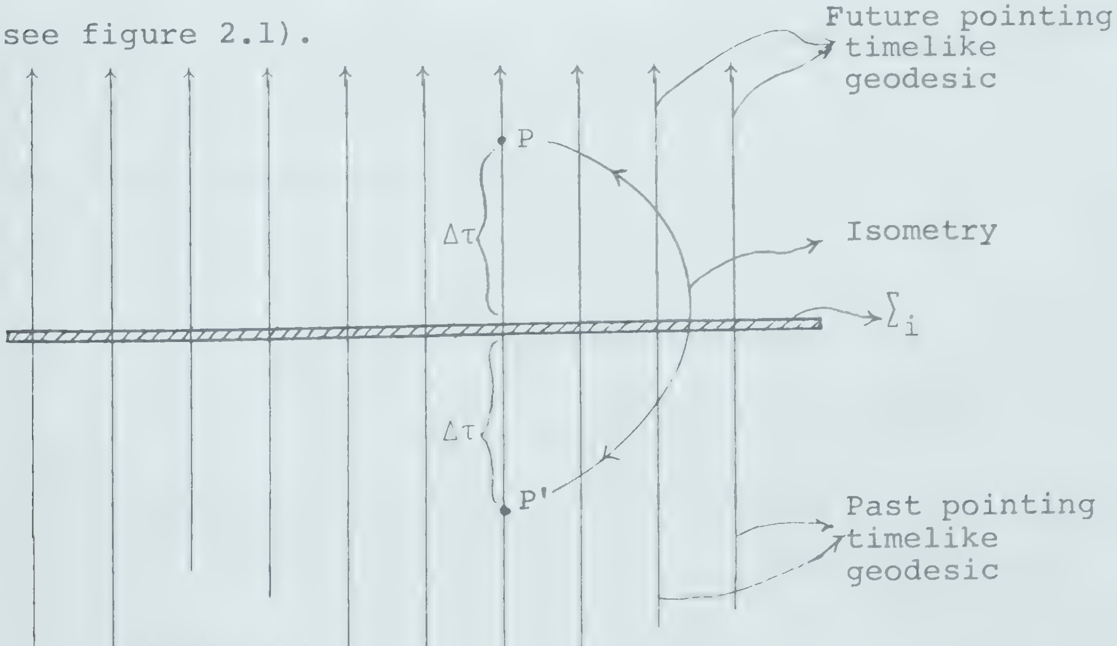

figure 2.1
A time-symmetric spacetime

Under the time-symmetric isometry, the spacetime 4-metric must have the following properties:

$$\left.\begin{aligned} {}^{4}g_{ij}(x^i,x^4) &= {}^{4}g_{ij}(x^i,-x^4) \\ {}^{(4)}g_{i4}(x^i,x^4) &= -\,{}^{4}g_{i4}(x^i,-x^4) \\ {}^{(4)}g_{44}(x^i,x^4) &= g_{44}(x^i,-x^4) \end{aligned}\right\} \quad . \tag{28}$$

To see how one arrives at such properties, consider an infinitesimal line element ds^2 in the future region of $\sum_i$. In terms of components of the metric tensor, the line element is

$$ds^2_{(f)} = g_{ij}(x^i,x^4)dx^i dx^j + 2g_{i4}(x^i,x^4)dx^i dx^4 + g_{44}(x^i,x^4) \times (dx^4)^2 \quad . \tag{29}$$

The isometry maps the future line element to a past line element with exactly the same magnitude. The spatial coordinate of the metric tensor remains unchanged while the time coordinate, i.e. x^4 reverses its direction after the mapping. Hence, the past line element has the form

$$\begin{aligned} ds^2_{(p)} &= g_{ij}(x^i,-x^4)dx^i dx^j + 2g_{4i}(x^i,-x^4)dx^i d(-x^4) \\ &\qquad + g_{44}(x^i,-x^4)d(-x^4)\cdot d(-x^4) \\ &= g_{ij}(x^i,-x^4)dx^i dx^j - 2g_{i4}(x^i,-x^4)dx^i dx^4 \\ &\qquad + g_{44}(x^i,-x^4)(dx^4)^2. \end{aligned} \tag{30}$$

Since the magnitude is invariant under the isometry, the two line elements must be identical, i.e.

$$ds^2_{(f)} = ds^2_{(p)} \quad .$$

Equation (28) follows when one compares the coefficients.

Two properties of g_{ij} are obvious from (28), namely,

(1) g_{ij} is an even function of x^4,

(2) g_{ij} is independent of x^4 on the initial spacelike hypersurface $\sum_i : t = 0$.

An immediate result that follows is that on $\sum_i : t = 0$,

$$\left.\frac{\partial g_{ij}}{\partial x^4}\right|_{\sum_i} = 0 \quad . \tag{31}$$

Also from equation (28), one sees that at $x^4 = 0$, that is on $\sum_i$,

$$g_{i4}(x^i, x^4{=}0) = - g_{i4}(x^i, -x^4{=}0) \quad .$$

This is equivalent to

$$g_{i4} = 0 \quad \text{on} \quad \sum_i \quad . \tag{32}$$

This is hardly surprising as the four degrees of freedom in Einstein field equation enable one to choose a suitable coordinate in which g_{i4} is identically zero everywhere in the spacetime manifold. However, one must realize that condition (32) is a condition imposed by

the physical nature (i.e. time-symmetry) of the space-time rather than the covariant nature of Einstein field of equations.

Let us now proceed to investigate how these two conditions, viz.,

$$\text{on } \Sigma_i : \quad \frac{\partial g_{ij}}{\partial x^4} = 0 \tag{31}$$

and

$$g_{i4} = 0 \tag{32}$$

simplify the constraint equations.

In the language of extrinsic curvature, the initial constraint equations are

$$\tfrac{1}{2}R - \tfrac{1}{2}\left(\frac{1}{\underline{n}\cdot\underline{n}}\right)[(\mathrm{Tr}\ K)^2 - \mathrm{Tr}(K^2)] = +\kappa T_4{}^4 \tag{33}$$

and

$$\left(\frac{1}{\underline{n}\cdot\underline{n}}\right)[K_i{}^j{}_{|j} - (\mathrm{Tr}\ K)_{|i}] = -\kappa T_i{}^4 \tag{34}$$

where $\underline{n}$ is the unit vector orthogonal to Σ_i

$$\underline{n}\cdot\underline{n} = \begin{cases} +1 & \text{if } \underline{n} \text{ is spacelike} \\ -1 & \text{if } \underline{n} \text{ is timelike} \end{cases}$$

and $\quad \mathrm{Tr}\ K \equiv K_i{}^i \quad$ and $\quad \mathrm{Tr}(K^2) \equiv K_i{}^j K_j{}^i$.

The extrinsic curvature K_{ij} is defined by

$$K_{ij} = \frac{1}{2N}\left[N_{i|j} + N_{j|i} - \frac{\partial g_{ij}}{\partial x^4}\right] \tag{35}$$

where N and N_i are the lapse and shift function respectively, and are related to the metric tensor by the equation

$$\begin{pmatrix} g_{ij} & g_{i4} \\ g_{4j} & g_{44} \end{pmatrix} = \begin{pmatrix} g_{ij} & N_i \\ N_j & N_i N^i - N^2 \end{pmatrix} . \tag{36}$$

This relation, with conditions (31) and (32), shows that

$$K_{ij} = 0 \quad \text{everywhere on } \textstyle\sum_i . \tag{37}$$

As Israel (1966) pointed out the extrinsic curvature measures the way a hypersurface "bends" with respect to the imbedded spacetime manifold. Equation (37) then says that the time-symmetric spacelike hypersurface is imbedded in a "flat" manner, that is, it does not "bend".

With this 20/20 hindsight, one can arrive at equation (37) with a more elegant argument. The time-symmetric isometry maps the unit vector $\underline{n}$ on $\sum_i$ into $-\underline{n}$ but preserves the orthogonality. One recalls that extrinsic curvature is defined by

$$\frac{\partial \underline{n}}{\partial \xi^i} = K_i{}^j \underline{e}_j \quad .$$

The isometry maps this into

$$\frac{\partial \underline{n}}{\partial \xi^i} = - K_i{}^j \underline{e}_j \quad ;$$

hence it implies, $K_i{}^j = - K_i{}^j$,

or $\quad K_{ij} = 0 \quad$ everywhere on $\sum_i$.

With the extrinsic curvature identically zero everywhere on $\sum_i$, the second constraint equation

$$\frac{1}{n \cdot n} [K_i{}^j{}_{|j} - (\text{Tr } K)_{|i}] = - \kappa T_i{}^4$$

imposes a time-symmetry condition on the sources, and the first constraint equation reduces to

$$\tfrac{1}{2}\, {}^3R = \kappa T_4{}^4 \quad . \tag{38}$$

To facilitate calculation, we concentrate from now on the vacuum case, i.e.

$$T_4{}^4 \equiv 0 \ .$$

This reduces the constraint to

$$^3R = 0 \ . \tag{39}$$

Even with a simple equation like $^3R = 0$, the complexity of R still defies detailed treatment. Without losing much generality, we further assume that $\sum_i$ is conformally flat. The assumption drastically reduces the complexity of R.

Consider two Riemannian manifolds V and $\bar{V}$ with metrics $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ respectively. The manifolds V and $\bar{V}$ are said to be conformal iff the metrics are related by

$$\bar{g}_{\mu\nu} = e^{2\sigma} g_{\mu\nu} \tag{40}$$

where σ is any function of x^α. Thus a spacetime is conformally flat if its line element obeys

$$ds^2 = e^{2\sigma}(ds^2)_{flat} \tag{41}$$

where $(ds^2)_{flat}$ is the line element of a flat spacetime.

It is well-known that the Riemannian scalars R and $\bar{R}$ of the conformal spaces are related by

$$\bar{R} = e^{-2\sigma}[R + 2(n-1)\Delta_2\sigma + (n-1)(n-2)\Delta_1\sigma] \tag{42}$$

where n is the dimension of V, and

$$\left.\begin{aligned} \Delta_1\sigma &\equiv g^{\mu\nu}\sigma_{,\mu}\sigma_{,\nu} \\ \Delta_2\sigma &\equiv g^{\mu\nu}(\sigma_{,\mu})_{|\nu} \\ &= g^{\mu\nu}[\sigma_{,\mu\nu} - \delta_{,\alpha}\Gamma^{\alpha}_{\mu\nu}]. \end{aligned}\right\} \tag{43}$$

Instead of using $e^{2\sigma}$ as the conformal factor, it is sometimes more convenient to express it as ψ^4. That is, instead of (40), one writes

$$\bar{g}_{\mu\nu} = \psi^4 g_{\mu\nu} \quad . \tag{44}$$

Comparing (40) and (44), one gets

$$\sigma = 2 \ln \psi \quad .$$

Substitute this into equations (43), one gets

$$\Delta_1\sigma = 4g^{\mu\nu}(\ln\psi)_{,\mu}\cdot(\ln\psi)_{,\nu}$$

$$= 4g^{\mu\nu}\psi^{-2}(\psi_{,\mu})(\psi_{,\nu}) \tag{45}$$

and

$$\Delta_2\sigma = 2g^{\mu\nu}[(\ln\psi)_{,\mu\nu} - (\ln\psi)_{,\alpha}\Gamma^{\alpha}_{\mu\nu}]$$

$$= 2g^{\mu\nu}[\frac{\psi_{,\mu\nu}}{\psi} - \frac{\psi_{,\mu}\psi_{,\nu}}{\psi^2} - \frac{\psi_{,\alpha}}{\psi}\Gamma^{\alpha}_{\mu\nu}]$$

$$= 2g^{\mu\nu}[\frac{1}{\psi}(\psi_{,\mu})_{|\nu} - \frac{\psi_{,\mu}\psi_{,\nu}}{\psi^2}] \quad . \tag{46}$$

Thus, in terms of ψ, equation (42) becomes

$$\bar{R} = \psi^{-4}[R + 4(n-1)g^{\mu\nu}\{\frac{1}{\psi}(\psi_{,\mu})_{|\nu}\} + 4(n-1)(n-3)g^{\mu\nu}\frac{1}{\psi^2}\psi_{,\mu}\psi_{,\nu}]. \tag{47}$$

For a conformally flat spacelike hypersurface $\sum_i$, one has

$$\bar{g}_{ij} = \psi^4(g_{ij})_{flat}$$

and

$$n = 3 \ .$$

This reduces (47) to

$$\bar{R} = \psi^{-4}[R_{flat} + 8(g^{\mu\nu}{}_{flat})\{\frac{1}{\psi}(\psi_{,\mu})_{|\nu}\}] \ . \tag{48}$$

But $R_{flat} \equiv 0$; thus we arrive at

$$\bar{R} = 8\frac{1}{\psi^5}\nabla^2\psi \tag{49}$$

where ∇^2 is the Laplacian operator of the flat space.

To sum up, the initial constraint equation for a time-symmetric spacetime is

$$\tfrac{1}{2}R = -\kappa T_4{}^4 \quad .$$

If $\sum_i$ is conformally flat, this is further reduced to

$$4\psi^{-5}\nabla^2\psi = -\kappa T_4{}^4 \tag{50}$$

where ψ is the conformal factor. For a vacuum spacetime, the constraint is simply the familiar Laplace equation

$$\nabla^2\psi = 0 \ . \tag{51}$$

CHAPTER 3

PERTURBATION OF THE MINIMAL SURFACE IN THE TIME-SYMMETRIC IVP

I. Introduction

It is a long established fact, first by Penrose (1965) and later generalized by Penrose-Hawking (1970), that under reasonable assumptions about spacetime, singularity is an inevitable end-result of a stellar object with mass greater than a few solar mass† undergoing gravitational collapse after its nuclear fuel has been exhausted. However, the question whether all such singularities are shielded from external observers at future null infinity remains unsettled. Different methods have since been contrived either (1) to construct a singularity that is visible, hence the name

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† Rhodes and Ruffini (1972) placed the upper limit of a stable neutron star at 3.2 solar mass. Other authors using various equations of state, arrived at different upper limits. For example, Oppenheimer (1939), 0.7 solar mass; Cameron (1970), 2.4 solar mass; Hartle (1972), 1.4; Harrison et al., (1965), 1.2 etc. However, Rhodes (1971) showed that no stable cold stellar configuration is possible at stellar mass exceeding 5 solar mass. Beyond this, collapse is inevitable.

"naked singularity", to an observer at infinity (Seifert et al., 1973, 1974) or (2) to arrive at some contradiction to known laws whose validity based on the assumption that no naked singularity exists (Penrose, 1971, Gibbons,1972, Hawking, 1973). The inability of these methods to obtain a naked singularity leads to what is popularly known as the "Penrose conjecture of cosmic censor" which states that

> "There exists a 'cosmic censor' which forbids the appearance of naked singularities, clothing each one in an absolute event horizon."
>
> Penrose (1969).

A definite proof of this conjecture is still non-existing.

The basic idea behind Gibbons' effort is trying to reach a contradiction with Hawking's (1972a) second law of Black Hole dynamics which states that in a future asymptotically predictable spacetime, which in heuristic terms means a spacetime with no "naked singularities", the total area of all event horizons can never decrease. Failing to reach a contradiction, Gibbons obtained, instead, an upper limit of the amount of energy that can be carried away by radiation, both electromagnetic and gravitational, during black holes coalition.

In this chapter, we elaborate on Gibbons' method to see if one can reach a contradiction to Hawking's

second law by concentrating on the effect of small perturbation on the area of the event horizon of a single black hole. Not surprisingly, the result reaffirms the validity of Penrose's conjecture. Hence, it seems exceedingly unlikely that such method can be used to contradict the censorship conjecture. We obtain, as Gibbons did, an upper limit of radiation reaching null infinity $\mathcal{I}^{\dagger}$ in terms of the perturbation parameter ε.

II. Event Horizon, Minimal Surface and A Theorem

Event Horizon

In the Schwarzschild coordinate, the metric of the Schwarzschild geometry is given by

$$ds^2 = \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\Omega^2 - \left(1 - \frac{2m}{r}\right) dt^2 . \tag{1}$$

From this equation, it is easy to see that the event horizon, that is the surface of the black hole, is evidently the surface

$$r = 2m .$$

But in coordinate system other than the Schwarzschild's things are not so apparent since there do not exist distinctive time and spatial coordinates like those in (1). To avoid confusions which arise from the deployment of a particular coordinate system, Penrose and Hawking developed a general and precise definition of

event horizon that is both coordinate-independent and theory-independent in that it does not depend on the outcome of general relativity.

Various experiments have shown that the trajectory of light rays is "bent" by the gravitational field it passes through, and that the degree of bending depends on the intensity of the gravitational field. It is then reasonable to expect light rays to be "trapped" in an extremely strong gravitational field, like that of a star going through the final stage of collapse. This leads to the concept of "trapped surfaces" which, first conceived by Penrose (1965) in his work on singularities, is essential to the idea of event horizon. Imagine a massive, transparent star with a light source at the center emitting a series of flashes of light. For simplicity's sake, assume perfect sphericity. Before, and at the early stage of, the collapse, the spherical wavefront of the light flash is able to expand continuously, though the rate of expansion is slightly retarded by the gravitational field, so that after a finite time interval the light sphere will emerge from the star and escapes to null infinity, $\mathcal{J}^{\dagger}$ (for a definition of $\mathcal{J}^{\dagger}$ and other notations, see Gravitation). However, as the collapse continues and the gravitational field intensifies, the rate of expansion, the divergence, of the

wavefront decreases due to the progressively stronger retardation until at a critical size of the star, the wavefront will emerge from the star with zero divergence. The area of this wavefront will then remain constant and can never reach $\mathcal{J}^{\dagger}$. As the field further intensifies due to continued collapse, the divergence of the emerging wavefront will acquire a negative value. In other words, the emerging wavefront will have an ever decreasing area until it hits the singularity with a zero area. Such outward-pointing null surfaces (i.e. the wavefronts) with ever decreasing area are called "trapped surfaces".

In more technical terms, "trapped surfaces" are compact, spacelike 2-surfaces whose outgoing, future-pointing, orthogonal (to the 2-surface) null geodesics converge at every point on the surface.

Evidently, it is impossible to transmit information from within the trapped surface to an observer at infinity. Thus, the existence of a trapped surface implies that there is a region in the spacetime within which it is impossible to reach future infinity via any non-spacelike curve. The boundary of this cut-off region is the light sphere that avoids both hitting the singularity, i.e. converging, or reaching future null infinity, $\mathcal{J}^{\dagger}$. This boundary then is the event horizon.

In Penrose-Hawking's notation, the event horizon is written as $\dot{J}(\mathscr{I}^{\dagger})$, that is, it is the boundary of the domain that can reach $\mathscr{I}^{\dagger}$ with a future-pointing causal curve.

A side-remark: The formation of the trapped surface, hence the formation of the event horizon and singularity, does not depend on the spherical symmetry we assumed at the beginning. A small deviation from spherical symmetry does not change the fact that an outgoing wavefront can be made to converge with a strong gravitational field. This is due to the stability of the Einstein equation.

Minimal Surface

Before we give the definition of minimal surface, let us first introduce the concept of extremal surface, or more generally, the extremal hypersurface.

Consider a manifold V_n with dimension n and a submanifold V_m with dimension m $(m < n)$. Let the submanifold V_m be defined by the equation

$$x^{\alpha} = f^{\alpha}(\bar{x}^{i}) \qquad (2)$$

where x^{α} are coordinates of V_n, $\bar{x}^{i}$ are coordinates of V_m; and they are not necessarily identical in the region of V_m.

(Note: In this part dealing with minimal surface, Greek indices run from 1 to n, and Latin indices run from 1 to m, i.e.

$$\alpha, \beta, \gamma, \ldots. = 1, 2, \ldots, n$$

$$i, j, k, \ldots. = 1, 2, \ldots, m\ )$$

Suppose there exists a closed hypersurface V_{m-1} in V_m enclosing a region, called D_m, of V_m. Now consider the integral taken over the volume D_m,

$$I = \int_{D_m} \mathcal{L}(x^\alpha, x^\alpha{}_{,i}, x^i)\ dx^1 dx^2 \ldots dx^m \tag{3}$$

where $\mathcal{L}$ is some function of x^α, $x^\alpha{}_{,i}$ and $\bar{x}^i$ only.

If the surface (eq. 2) is perturbed by an infinitesimal amount,

$$x^\alpha = f^\alpha(\bar{x}^i) + \varepsilon\omega^\alpha(\bar{x}^i) \tag{4}$$

where ε is an infinitesimal quantity and $\omega^\alpha(\bar{x}^i)$ an arbitrary function of $\bar{x}^i$ that vanishes on V_{m-1}, i.e.

$$\omega^\alpha(\bar{x}^i) = 0 \quad \text{for all points on} \quad V_{m-1}\ , \tag{5}$$

the change of the integral of (3) is, up to the first order of ε,

$$\delta I = \varepsilon \int_{D_m} \{\frac{\partial \mathcal{L}}{\partial f^\alpha}\ \omega^\alpha + \frac{\partial \mathcal{L}}{\partial (f^\alpha{}_{,i})}\ \omega^\alpha{}_{,i}\}\ dx^1 dx^2 \ldots dx^n \ . \tag{6}$$

With equation (5), an integration by parts gives

$$\delta I = \varepsilon \int \{ \frac{\partial \mathcal{L}}{\partial f^{\alpha}} - \frac{\partial}{\partial \bar{x}^{i}} (\frac{\partial \mathcal{L}}{\partial (f^{\alpha}_{\ ,i})}) \omega^{\alpha} \cdot dx^{1} dx^{2} \ldots dx^{m} \ . \tag{7}$$

In order that the integral of (3) be stationary with respect to the variation of the type in equation (5), we must have

$$\delta I = 0$$

or

$$\frac{\partial \mathcal{L}}{\partial f^{\alpha}} - \frac{\partial}{\partial \bar{x}^{i}} (\frac{\partial \mathcal{L}}{\partial (f^{\alpha}_{\ ,i})}) = 0 \ . \tag{8}$$

This is, of course, the well-known Euler-Lagrange equation.

The area element of a 2-surface in a Euclidean 3-space is

$$dA = (EG - F^{2}) dx^{1} dx^{2} \tag{9}$$

where E, F, G are coefficients of the 2-metric,

$$ds^{2} = E(dx^{1})^{2} + 2F\ dx^{1} dx^{2} + G(dx^{2})^{2} \ . \tag{10}$$

Likewise, the "area" element of a hypersurface can be defined as

$$\sqrt{g}\ dx^{1} \ldots dx^{m}$$

where g is the determinant of the metric of the hypersurface.

If we take $\mathcal{L}$ in equation (3) to be $\sqrt{g}$, then the integral

$$I = \int_{D_m} \sqrt{g}\, dx^1 dx^2 \ldots dx^m$$

gives the "area" of the hypersurface V_m in V_n. The hypersurface V_m is then said to be extremal if the Euler-Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial f^\alpha} - \frac{\partial}{\partial \bar{x}^i}\left(\frac{\partial\mathcal{L}}{\partial (f^\alpha{}_{,i})}\right) = 0$$

with $\mathcal{L} = \sqrt{g}$ is satisfied.

Having introduced the idea of extremal surface, the minimal surface can be defined as a surface which is extremal, that is, its first variation vanishes, but whose second variation is always positive, i.e.

$$\delta^2 I > 0 \; .$$

<u>Lemma and Theorems</u>

We state below a lemma and a theorem by Gibbons (1972). The details of the proofs are omitted as they are rather straightforward.

Lemma: Consider a Riemannian manifold V with twice differentiable metric. Let Σ be a submanifold of V.

If there exists an isometry $\phi: V \to V$ that leaves every point of Σ point-wise fixed, then Σ is a totally geodesic manifold, and hence an extremal hypersurface.

Remarks: (1) A totally geodesic hypersurface is a hypersurface on which all geodesics of the hypersurface are also geodesics of the enveloping manifold. (2) The proof relies on that if there exists a geodesic that lies partially on Σ, the isometry will map this geodesic into two different parts. There will then be a point on Σ that has two distinct geodesics with the same initial tangent. This is impossible. (3) In a theorem by Eisenhart (1966), he showed that a totally geodesic hypersurface is necessarily extremal. (They both have vanishing extrinsic curvature.)

Theorem 1: A compact, orientable two-surface S, which is minimal, lying in a three-surface S having non-negative Ricci tensor R_{ij}, is homeomorphic to a 2-sphere.

Remarks: (1) Since S is minimal, the second variation of the area must be positive, i.e. $d^2A/dt^2 > 0$ where t is the normal direction of S.

(2) It can be shown that

$$\frac{d^2A}{dt^2} = \int\{-\frac{{}^3R}{2} - \Sigma^2 + K\}dA$$

where R is the Ricci scalar,

$$\Sigma^2 \equiv \frac{1}{2}\Sigma_{ij}\Sigma^{ij}$$

where Σ_{ij} is the trace-free part of the second fundamental form, and K is the Gaussian curvature of S.

(3) In order that remark (1) be satisfied, we must have $\int KdA > 0$. By the Gauss-Bonnet theorem

$$\int K dA = 4\pi(1 - P)$$

where P is the "genus", or the number of handle the surface has. And P = 0 for sphere; = 1 for torus, etc. Therefore the type of surface that allows

$$\int K dA > 0$$

is the sphere with P = 0.

III. Perturbation of the Schwarzschild Initial Data

The perturbation is carried out on a time-symmetric, conformally flat, spacelike hypersurface $\sum_i$. On such a hypersurface, the initial constraint equation reduces to

$$4\psi^{-5}\nabla^2\psi = - \kappa T_4{}^4 ,$$

and, in the case of a vacuum field,

$$\nabla^2\psi = 0 . \tag{11}$$

The Schwarzschild solution, being a vacuum solution to the complete Einstein field equation, is also a solution of (11). In isotropic coordinates, it reads

$$ds^2 = (1 + \frac{m}{2r})^4 [dr^2 + r^2 d\Omega^2] - \frac{(1 - \frac{m}{2r})^2}{(1 + \frac{m}{2r})^2} dt^2 . \tag{12}$$

This gives the conformal factor

$$\psi = (1 + \frac{m}{2r}) . \tag{13}$$

The event horizon as well-known has radius and area $r = m/2$, $A = 16\pi m^2$ respectively.

The event horizon in a time-symmetric spacetime is the minimal surface. Before giving a rigorous proof, we take a look at the Kruskal diagram to illustrate the point.

In its original form, the Schwarzschild metric reads

$$ds^2 = \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\Omega^2 - (1 - \frac{2m}{r})\, dt^2 \quad . \tag{14}$$

This can be transformed into the Kruskal (1960)-Szekeres (1960) form

$$ds^2 = (\frac{32m^3}{r})\, e^{-r/2m}(du^2 - dv^2) + r^2 d\Omega^2 \tag{15}$$

by the transformation

$$\left.\begin{aligned} u &= (\frac{r}{2m} - 1)^{\frac{1}{2}}\, e^{r/4m} \cosh(\frac{t}{4m}) \\ v &= (\frac{r}{2m} - 1)^{\frac{1}{2}}\, e^{r/4m} \sinh(\frac{t}{4m}) \end{aligned}\right\} \quad \text{for } r > 2m$$

$$\left.\begin{aligned} u &= (\frac{r}{2m} - 1)^{\frac{1}{2}}\, e^{r/4m} \sinh(\frac{t}{4m}) \\ v &= (\frac{r}{2m} - 1)^{\frac{1}{2}}\, e^{r/4m} \cosh(\frac{t}{4m}) \end{aligned}\right\} \quad \text{for } r < 2m$$

Or, inversely,

$$\left(\frac{r}{2m} - 1\right)e^{r/2m} = u^2 - v^2$$

$$t = \begin{cases} 4m\tanh^{-1}(v/u) & \text{for } r > 2m \\ 4m\tanh^{-1}(u/v) & \text{for } r < 2m \end{cases}$$

In terms of the (u,v) coordinates, the complete Schwarzschild with (θ,ϕ) suppressed are shown in figure 3.1. Each point in the diagram then represents a 2-sphere in the spacetime manifold.

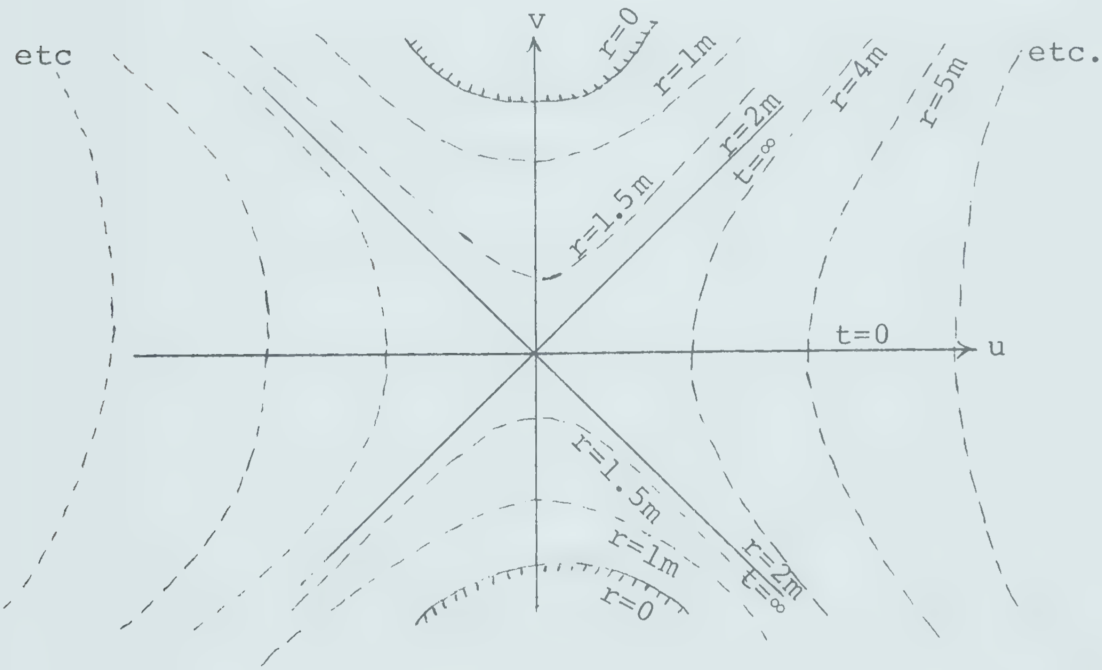

figure 3.1

Schwarzschild geometry in Kruskal-Szekeres coordinates

With this diagram, the time-symmetry of the Schwarzschild geometry is self-evident. The time-symmetric, spacelike hypersurface Σ is the u axis. If one travels along the u axis from right to left, one passes through points

(2-spheres) with decreasing r (or decreasing area), until r = 2m (the event horizon!) is reached. Further left, one again encounters points with increasing r. Thus, r = 2m is the minimal sphere $\sum_i$ would admit. This sphere as well-known is the Schwarzschild event horizon with area $16\pi m^2$.

In the following, we give the proof that the event horizon in the time-symmetric hypersurface must be the minimal surface. A lemma is needed to complete the proof.

Lemma: If V_m is a submanifold of V_n, described by parametric equations

$$x^\alpha = x^\alpha(y^i)$$

and having metrics, respectively,

$$ds^2 = h_{ij}dy^i dy^j \qquad (i,j,\ldots = 1,2,\ldots m)$$

and $$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta \qquad (\alpha,\beta,\ldots = 1,2,\ldots n) ,$$

let A^α be a vector field tangent to V_m, then the projection of the covariant derivative of A^α with respect to $g_{\alpha\beta}$ onto V_m is equal to its covariant derivative with respect to h_{ij}.

Symbolically,

$$({}^{(n)}\nabla_\beta A_\alpha) e^\alpha_{(i)} e^\beta_{(j)} = {}^{(m)}\nabla_i A_j$$

where

$$e^{\alpha}_{(i)} \equiv \frac{\partial x^{\alpha}}{\partial y^{i}} \quad .$$

Proof: Consider a line element in V_m, in terms of h_{ij}, this is

$$ds^2 = h_{ij}dy^i dy^j \;;$$

in terms of $g_{\alpha\beta}$, it is

$$ds^2 = g_{\alpha\beta}dx^{\alpha}dx^{\beta} \;.$$

Therefore, it must have

$$g_{\alpha\beta}dx^{\alpha}dx^{\beta} = h_{ij}dy^i dy^j$$

or, $$g_{\alpha\beta}\,\frac{\partial x^{\alpha}}{\partial y^{i}}\,\frac{\partial x^{\beta}}{\partial y^{j}} = h_{ij} \quad . \tag{16}$$

Since $\partial x^{\alpha}/\partial y^{i}$ form a complete set of m vectors tangent to V_m, they can be used as a basis for V_m. The contravariant basis can then be defined,

$$e^{(j)\alpha} \equiv h^{ij}e_{i}^{(\alpha)} = h^{ij}\,\frac{\partial x^{\alpha}}{\partial y^{i}}$$

Similarly,

$$e^{(j)}{}_{\beta} = g_{\alpha\beta}e^{(j)\alpha} \quad .$$

Since the vector field $\underline{A}$ is tangential to V_m, we can write

$$A_{\alpha} = A_i e^{(i)}{}_{\alpha} \quad .$$

The covariant derivative of A_α with respect to $g_{\alpha\beta}$ is ${}^n\nabla_\beta A_\alpha$; its projection onto V_m is

$$({}^{(n)}\nabla_\beta A_\alpha)e^\alpha_{(i)}e^\beta_{(j)} = [{}^{(n)}\nabla_\beta(A_\alpha e^\alpha_{(i)}) - A_\alpha\nabla_\beta e^\alpha_{(i)}]e^\beta_{(j)}$$

$$= [\nabla_\beta A_i - A_k e^{(k)}_\alpha \nabla_\beta e^\alpha_{(i)}]e^\beta_{(j)} \quad . \tag{17}$$

The component of $\underline{A}$ with respect to y^i is invariant under a coordinate transformation of x^α's; thus A_i behaves like a scalar in x^α coordinates. Therefore

$$({}^{(n)}\nabla_\beta A_i)e^\beta_j = (\partial_\beta A_i)e^\beta_{(j)} = A_{i,j} \quad . \tag{18}$$

Define a quantity $\quad {}^*\Gamma^k_{ij} \equiv e^{(k)}{}_\alpha(\nabla_\beta e^\alpha_{(i)})e^\beta_{(j)} \quad .$ (19)

With (18) and (19), (17) becomes

$$[{}^{(n)}\nabla_\beta A\ ]e^\alpha_{(i)}e^\beta_{(j)} = A_{i,j} - A_k {}^*\Gamma^k_{ij} \quad . \tag{20}$$

If we can show that ${}^*\Gamma^k_{ij}$ defined by (19) is indeed the Christoffel symbol for h_{ij}, our proof is complete. To achieve this, recall that the covariant derivative of h_{ij} vanishes, i.e.

$${}^{(n)}\nabla_k(h_{ij}) = 0 \ .$$

This defines the Christoffel symbol for h_{ij}, i.e.

$$\partial_k h_{ij} = h_{i\ell}\Gamma^\ell_{kj} + h_{j\ell}\Gamma^\ell_{ki} \quad . \tag{21}$$

The metric of V_m, h_{ij}, is completely determined by the intrinsic coordinates of V_m, and hence behaves like a scalar with respect to x^α, the coordinates of V_n. We therefore can write,

$$\partial_\alpha (h_{ij}) = {}^{(n)}\nabla_\alpha (h_{ij}) \ .$$

Keeping this in mind, we have

$$\begin{aligned}
\partial_k (h_{ij}) &= e^\alpha_{(k)} \partial_\alpha (h_{ij}) = e^\alpha_{(k)} \cdot {}^{(n)}\nabla_\alpha [e^\beta_{(i)} e_{\beta(j)}] \\
&= e^\alpha_{(k)} \cdot e^\beta_{(i)} \cdot {}^{(n)}\nabla_\alpha e_{\beta(j)} + e^\alpha_{(k)} \cdot e_{\beta(j)} \cdot {}^{(n)}\nabla_\alpha e^\beta_{(i)} \\
&= h_{i\ell} \cdot e^\alpha_{(k)} \cdot e^{(\ell)\beta} \cdot {}^{(n)}\nabla_\alpha e_{\beta(j)} + h_{j\ell} \cdot e^\alpha_{(k)} e^{(\ell)}_\beta \, {}^{(n)}\nabla_\alpha e^\beta_{(i)} .
\end{aligned} \tag{22}$$

And by definition of (19), this is just

$$\partial_k (h_{ij}) = h_{i\ell} {}^*\Gamma^\ell_{kj} + h_{j\ell} {}^*\Gamma^\ell_{ik} \ . \tag{23}$$

A comparison of (21) and (23) shows that ${}^*\Gamma^\ell_{ik}$ as defined by (19) is indeed the Christoffel symbol of h_{ij}, and hence the proof.

Q.E.D.

Consider a compact orientable 2-surface F imbedded in $\sum_i$. Let $\underline{n}$ be the unit timelike vector orthogonal to $\sum_i$, and $\underline{s}$ be spacelike unit vector parallel to $\sum_i$ but orthogonal to F. The future null vector $\underline{\ell}$ is given by $\ell^\alpha = n^\alpha + s^\alpha$.

Using Sachs (1959) method, together with ℓ^α one can choose three more null vectors, one real and two complex, to form a null tetrad. First, pick a future null vector k^α not parallel to ℓ^α such that

$$k^\alpha \ell_\alpha = -1 \ .$$

These two real null vectors form a timelike 2-flat. For the (spacelike) 2-surface, i.e. F, orthogonal to this, choose two complex null vectors m^α, $\bar{m}^\alpha$ parallel to F such that

$$m_\alpha m^\alpha = 0 \ , \qquad m_\alpha \bar{m}^\alpha = 1 \ .$$

The four null vectors then span the spacetime.

The completeness relation then gives

$$g_{\alpha\beta} = m_\alpha \bar{m}_\beta + m_\beta \bar{m}_\alpha - \ell_\alpha k_\beta - \ell_\beta k_\alpha \quad . \tag{24}$$

Now, instead of using m^α, $\bar{m}^\alpha$ for the spacelike 2-surface, one can use $e^\alpha_{(A)}$ as basis for F. Equation (20) can then be written as

$$g^{\alpha\beta} = -\ell^\alpha k^\beta - \ell^\beta k^\alpha + g^{AB} e^\alpha_{(A)} e^\beta_{(B)} \quad . \qquad (A,B,..=1,2)$$

Consider the 2-surface F, the convergence of future null geodesics orthogonal to F is given by

$$\rho \equiv g^{\alpha\beta} \ \ell_{\alpha|\beta} \quad .$$

If F is the marginally trapped surface, $\rho = 0$. This implies

$$0 = g^{\alpha\beta}\ell_{\alpha|\beta} = g^{AB}e^{\alpha}_{(A)}e^{\beta}_{(B)}\ell_{\alpha|\beta} \tag{25}$$

since $\ell_{\alpha|\beta}\ell^{\beta} = \ell_{\alpha|\beta}\ell^{\alpha} = 0$. Expressing ℓ_{α} explicitly, (25) becomes

$$g^{AB}e^{\alpha}_{(A)}e^{\beta}_{(B)}(n_{\alpha|\beta} + s_{\alpha|\beta}) = 0 \ . \tag{26}$$

The extrinsic curvature of $\sum_i$ imbedded in the spacetime manifold is defined by

$$K_{ab} \equiv n_{\alpha|\beta}e^{\alpha}_{(a)}e^{\beta}_{(b)} \ .$$

For F imbedded in $\sum_i$, the extrinsic curvature is

$$K^{(F)}_{AB} = ({}^3\nabla_b s_a)e^{a}_{(A)}e^{b}_{(B)} \ .$$

With this and the preceding lemma, equation (26) becomes

$$g^{AB}K_{AB} + g^{AB}K^{(F)}_{AB} = 0 \ . \tag{27}$$

It has been shown that for time-symmetric $\sum_i$, K_{ab} vanishes identically. Thus we are left with

$$g^{AB}K^{(F)}_{AB} = 0 \ .$$

By the properties† of extremal surface, this implies F is an extremal surface in $\sum_i$.

Besides being extremal, F is also a minimal surface since it is the only extremal surface. Outside F, one can enclose F in a succession of outer closed 2-surfaces. These 2-surfaces must have an area larger than F since the asymptotic flatness demands that the area of these surfaces be infinitely large as r tends to infinity. Hence, F must be the minimal surface. This is evident from the Kruskal diagram of fig. 4.1 in which F is represented by the intersection point of u,v axis.

To sum up, any marginally-trapped 2-surface in a time-symmetric hypersurface $\sum_i$ is a minimal surface. Conversely, any minimal surface F in $\sum_i$ is marginally trapped. For the static case, it is also the event

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† From the definition of extremal surface, one has

$$\frac{\partial\sqrt{g}}{\partial f^\alpha} - \frac{\partial}{\partial \bar{x}^i}\left(\frac{\partial\sqrt{g}}{\partial f^\alpha_{,i}}\right) = 0 .$$

Eisenhart (1966, Riemannian Geometry, p.178) showed that

$$\frac{\partial\sqrt{g}}{\partial f^\alpha} - \frac{\partial}{\partial \bar{x}^i}\left(\frac{\partial\sqrt{g}}{\partial f^\alpha_{,i}}\right) = \sqrt{g}\, g_{\alpha\beta} K_{ij} g^{ij} \xi^\beta .$$

Thus, the surface is extremal if $K_{ij}g^{ij} = 0$.

horizon. This theorem can easily be generalized to the (t,ϕ) reversible and axisymmetric hypersurface treated in Chapter 4.

With this theorem, we are ready to carry out the perturbation.

Consider a set of initial data on $\sum_i$ represented by

$$ {}^3ds^2_p = \psi_p{}^4 (dr^2 + r^2 d\Omega^2) \tag{28} $$

where

$$ \psi_p \equiv \psi_s + \varepsilon F(x,\theta,\phi), $$

ε being a numerical parameter and F, a function of (r,θ,ϕ) to be specified, and ψ_s is given by (13).

The constraint equation (11) requires that

$$ \nabla^2 \psi_p = 0 \ . $$

This implies that F must satisfy the Leplace equation

$$ \nabla^2 F = 0 \ . $$

A solution to this equation is

$$ F = \frac{P_\ell(\cos\ \theta)}{r^{\ell+1}} e^{im\phi} $$

where m, ℓ are integers and $P_\ell(\cos\ \theta)$ is the Legendre polynomial. Without losing generality, we choose F to be

$$F = \frac{P_\ell(\cos\theta)}{r^{\ell+1}} \quad .$$

With this, the perturbed Schwarzschild metric of (28) now looks

$$ds^2 = (1 + \frac{m}{2r} + \frac{\varepsilon P_\ell}{r^{\ell+1}})^4 \{dr^2 + r^2 d\Omega^2\} \quad . \tag{29}$$

Since the geometry of (29) differs from the Schwarzschild geometry, the minimal surface also changes. If for simplicity's sake, one considers only perturbations of infinitesimal order, i.e. $\varepsilon \to 0$, the minimal surface will change by an infinitesimal amount. Therefore, we can write

$$r_{min} = \frac{m}{2} + \varepsilon g_1(\theta) + \varepsilon^2 g_2(\theta) + \ldots \tag{30}$$

where $m/2$ is the radius of the unperturbed minimal surface.

In general, we write

$$r = f(\theta) \quad . \tag{31}$$

With (31), the metric now becomes

$$ds^2 = [1 + \frac{m}{2f} + \frac{P_\ell(\cos\theta)}{f^{\ell+1}}]^4 \{(f'^2+f^2)d\theta^2 + f^2\sin^2\theta \, d\phi^2\}$$

where

$$f' \equiv \frac{\partial f}{\partial \theta} \quad .$$

The area of the surface $r = f(\theta)$ is given by

$$A = \int [1 + \frac{m}{2f} + \frac{\varepsilon P_\ell}{f^{\ell+1}}]^4 (f'^2 + f^2)^{\frac{1}{2}} f \sin\theta \, d\theta \, d\phi \tag{32}$$

$$= 2\pi \int L \, d\theta$$

where we put

$$L \equiv \psi_p{}^4 (f'^2 + f^2)^{\frac{1}{2}} f \sin\theta \ , \tag{33}$$

and

$$\psi_p = [1 + \frac{m}{2f} + \frac{\varepsilon P_\ell}{f^{\ell+1}}] \ .$$

A variation of the area gives

$$\delta A = 2\pi \int \{\frac{\partial L}{\partial f} - \frac{d}{d\theta} \frac{\partial L}{\partial f'})\} \delta f \, d\theta \ .$$

For a minimal surface, its first variation must vanish. Hence, at r_{min}, we have

$$\frac{\partial L}{\partial f} - \frac{d}{d\theta} (\frac{\partial L}{\partial f'}) = 0 \ . \tag{34}$$

A simple differentiation of (33) gives

$$\frac{\partial L}{\partial f} = L \left(\frac{4 (\frac{\partial \psi_p}{\partial f})}{\psi_p} + \frac{f}{f'^2 + f^2} + \frac{1}{f} \right) \ ,$$

$$\frac{\partial L}{\partial f'} = L \{ \frac{f'}{f'^2 + f^2} \} \quad , \quad \text{etc.}$$

Further calculation reduces (34) to

$$4\psi_p^{-1}\,\dot{\psi}_p + (f^2+f'^2)^{-1}f + f^{-1}$$

$$- f'^2(f'^2+f^2)^{-1}\{4\psi_p^{-1}\dot{\psi}_p + f(f^2+f'^2)^{-1} + f^{-1}\}$$

$$- f'^2 f''(f'^2+f^2)^{-2} - \{\cot\theta - \varepsilon 4f^{-(\ell+1)}\cdot P'_\ell\cdot\sin\theta\}f'(f'^2+f^2)^{-1}$$

$$+ f''(f'^2+f^2)^{-1} + 2f'^2(f''+f)(f'^2+f^2)^{-2} = 0\ , \qquad (35)$$

evaluated at r_{min}, where $\dot{\psi}_p \equiv \partial\psi_p/\partial f$.

To determine the functions $g_1(\theta)$, $g_2(\theta)$, ..., we have to substitute r_{min}, or f_{min}, into (35). Partly to simplify calculation and mainly because the perturbation is of infinitesimal order, all terms of order ε^2 will be dropped. With this in mind, (35) reduces drastically to

$$4\psi_p^{-1}\dot{\psi}_p + 2f^{-1} - \cot\theta\cdot f'\cdot f^{-2} + f^{-2}f'' = 0\ . \qquad (36)$$

Explicitly this gives

$$f'' + f'\cot\theta - 2f + 4\ \frac{[k + (\ell+1)\varepsilon P_\ell f^{-\ell}]}{[1 + kf^{-1} + \varepsilon P_\ell f^{-(\ell+1)}]} = 0 \qquad (37)$$

where $k \equiv m/2$.

To further simplify (37), we substitute into (37),

$$f \simeq k + \varepsilon g_1$$

$$f' \simeq \varepsilon g'_1$$

$$f'' \simeq \varepsilon g''_1$$

$$f^{-(\ell+1)} \simeq k^{-(\ell+1)}[1-(\ell+1)\frac{\varepsilon g_1}{k}]$$

$$\psi_p^{-1} \simeq \frac{1}{2}[1+\frac{1}{2}\frac{\varepsilon g_1}{k}-\frac{1}{2}\frac{\varepsilon P_\ell}{k^{\ell+1}}]$$

and

$$[k+(\ell+1)f^{-\ell}\varepsilon P_\ell] \simeq [k+(\ell+1)k^{-\ell}\cdot\varepsilon P_\ell] \ .$$

After some calculation, the Euler-Lagrange equation of (13) reduces to a linear differential equation involving only $g_1(\theta)$,

$$g_1'' + g_1' \cot\theta - g_1 + 2(\ell+\tfrac{1}{2})\frac{P_\ell(\cos\theta)}{k^\ell} = 0 \ . \tag{38}$$

This equation looks baffling at first, however, a change of variables will convert it into a more familiar form.

Let $x = \cos\theta$, then

$$\cot\theta = \frac{x}{(1-x^2)^{\frac{1}{2}}} \quad , \qquad \frac{\partial x}{\partial\theta} = -\sin\theta = -(1-x^2)^{\frac{1}{2}} \ ,$$

and $$g_1'(\theta) = \frac{\partial g_1}{\partial\theta} = \frac{\partial g_1}{\partial x}\frac{\partial x}{\partial\theta} = -\dot{g}_1(1-x^2)^{\frac{1}{2}}$$

$$g_1''(\theta) \equiv \frac{\partial^2 g_1}{\partial\theta^2} = \frac{\partial}{\partial x}[-\dot{g}_1(1-x^2)^{\frac{1}{2}}]\cdot[-(1-x^2)^{\frac{1}{2}}]$$

$$\equiv \ddot{g}_1(1-x^2) - \dot{g}_1 x$$

where we used $\dot{g}_1 \equiv \partial g_1/\partial x$ and so on.

With this change of variable, equation (38) becomes

$$(1-x^2)\ddot{g}_1 - 2x\dot{g}_1 - g_1(x) + 2(\ell+\tfrac{1}{2})\frac{P_\ell(x)}{k^\ell} = 0. \quad (39)$$

In this form, it is easily recognized that the solution must be a Legendre polynomial with proper coefficient, i.e.

$$g_1 = AP_\ell(x) \quad (40)$$

where A is a constant to be determined.

To determine A, substitute (40) back to (39). This gives

$$(1-x^2)A\ddot{P}_\ell(x) - 2Ax\dot{P}_\ell(x) - AP_\ell(x) + 2(\ell+\tfrac{1}{2})\frac{P_\ell(x)}{k^\ell} = 0. \quad (41)$$

After rearranging, we have

$$(1-x^2)\ddot{P}_\ell - 2x\dot{P}_\ell + P_\ell[-1 + 2\frac{(\ell+\frac{1}{2})}{Ak^\ell}] = 0 .$$

Since P_ℓ satisfies the differential equation

$$(1-x^2)\ddot{P}_\ell - 2x\dot{P}_\ell + (\ell)(\ell+1)P_\ell = 0 ,$$

we must choose

$$\frac{2(\ell+\frac{1}{2})}{Ak^\ell} - 1 = \ell(\ell+1)$$

or

$$A = \frac{2(\ell+\frac{1}{2})}{k^\ell(\ell^2+\ell+1)} \quad . \quad (42)$$

Thus we arrive at the radius of the minimal surface in the perturbed Schwarzschild geometry. The radius is given by

$$r_{min} = k + \varepsilon \frac{2(\ell+\frac{1}{2})}{k^{\ell}(\ell^2+\ell+1)} P_{\ell}(\cos\theta) + O(\varepsilon^2) \qquad (42a)$$

where $k = m/2$. We are now all set to investigate how the area of the minimal surface is affected by this type of perturbation.

From equation (32), the area of the minimal surface is given by

$$A_{min} = 2\pi \int \psi_p^4 (f^2+f'^2)^{\frac{1}{2}} f \sin\theta \; d\theta \qquad (43)$$

evaluated at r_{min}.

When the area of the minimal surface is evaluated, terms involving ε^2 are retained. As r_{min} is only accurate up to order of ε, such practice needs justification. In the following, we will show that the $\varepsilon^2 g_2(\theta)$ term of r_{min} (cf equation (30)) enters the area through ε^3 or higher order terms. That is, $g_2(\theta)$ is not needed to evaluate the area up to ε^2 terms.

Consider a Lagrangian $L[x,y(x,\varepsilon), y'(x,\varepsilon),\varepsilon]$ and the action

$$I(\varepsilon) = \int_a^b L[x,y(x,\varepsilon), y'(x,\varepsilon),\varepsilon] \, dx \, .$$

Suppose $y(x,\varepsilon)$ can be expressed as a power series

$$y(x,\varepsilon) = y_o(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \varepsilon^3 y_3(x) + \ldots$$

and after evaluating the integral, the action can be expressed as

$$I(\varepsilon) = I_o + \varepsilon I_1 + \varepsilon^2 I_2 + \ldots \tag{44}$$

Differentiation with respect to ε gives

$$\dot{I}(\varepsilon) = \frac{\partial L}{\partial y'} \frac{dy}{d\varepsilon}\bigg|_a^b + \int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\right] \frac{\partial y}{\partial \varepsilon}\, dx + \int_a^b \frac{\partial L}{\partial \varepsilon}\, dx .$$

By virtue of the Euler-Lagrangian equation, the second term vanishes. We therefore get

$$\dot{I}(\varepsilon) = \frac{\partial L}{\partial y'} \frac{dy}{d\varepsilon}\bigg|_a^b + \int_a^b \frac{\partial L}{\partial \varepsilon}\, dx . \tag{45}$$

Similarly, the second derivative is given by

$$\ddot{I}(\varepsilon) = \frac{d}{d\varepsilon}\left[\frac{\partial L}{\partial y'} \frac{dy}{d\varepsilon}\right]\bigg|_a^b + \int_a^b \left[\frac{\partial^2 L}{\partial \varepsilon^2} + \frac{\partial^2 L}{\partial y \partial \varepsilon} \frac{\partial y}{\partial \varepsilon} + \frac{\partial^2 L}{\partial y' \partial \varepsilon} \frac{\partial y'}{\partial \varepsilon}\right] dx \tag{46}$$

but

$$\int_a^b \frac{\partial^2 L}{\partial y' \partial \varepsilon} \frac{\partial y'}{\partial \varepsilon}\, dx = \int_a^b \frac{\partial^2 L}{\partial y' \partial \varepsilon} \frac{d}{dx}\left(\frac{\partial y}{\partial \varepsilon}\right) dx$$

$$= \frac{\partial^2 L}{\partial y' \partial \varepsilon} \cdot \frac{\partial y}{\partial \varepsilon}\bigg|_a^b - \int_a^b \left(\frac{\partial y}{\partial \varepsilon}\right) \frac{d}{dx}\left(\frac{\partial^2 L}{\partial y' \partial \varepsilon}\right) dx . \tag{47}$$

The last term of (47) can be rewritten as

$$- \int_a^b \left(\frac{\partial y}{\partial \varepsilon}\right) \frac{\partial}{\partial \varepsilon} \left[\frac{d}{dx} \left(\frac{\partial L}{\partial y'}\right)\right] dx \ .$$

This, combined with the second term of the integral of (46), vanishes because of Euler-Lagrange equation. So we are left with

$$\ddot{I}(\varepsilon) = \frac{d}{d\varepsilon} \left[\frac{\partial L}{\partial y'} \frac{\partial y}{\partial \varepsilon}\right]\Bigg|_a^b + \frac{\partial^2 L}{\partial y' \partial \varepsilon} \frac{dy}{d\varepsilon}\Bigg|_a^b + \int_a^b \frac{\partial^2 L}{\partial \varepsilon^2} dx \ . \tag{48}$$

Twice differentiation of (44) gives

$$\ddot{I}(\varepsilon) = I_2 + \varepsilon I_1 + \ldots \tag{49}$$

If we can show I_2 to be independent of y_2, our assertion is justified. Look at (48), the only time y_2 is not multiplied by ε is in the term

$$\frac{\partial L}{\partial y'} \frac{d^2 y}{d\varepsilon^2}\Bigg|_a^b \ .$$

However, for the Lagrangian (33) we are considering $\partial L/\partial y' = 0$ at both limits of integration. Thus, all the y_2's are accompanied by ε at least. This then implies that y_2 appears only in I_3 or higher terms.

To evaluate the integral of (43), approximation procedures similar to those employed before are used, with the exception that ε^2 terms are retained. After

some lengthy calculations, we obtain,

$$A_m \doteq 2\pi \int_0^{\pi} [16 - \frac{32\varepsilon HP_\ell}{k} + \frac{32\varepsilon^2 AKP_\ell^2}{k^2} + \frac{24\varepsilon^2 H^2 P_\ell^2}{k^2}]$$

$$\times k[1 + \frac{\varepsilon AP_\ell}{k} + \frac{1}{2}\frac{\varepsilon^2 A^2 P_\ell'^2 \sin^2\theta}{k^2}]$$

$$\times k[1 + \frac{\varepsilon AP_\ell}{k}] \sin\theta \; d\theta \tag{50}$$

where $H \equiv [A - k^{-\ell}]$, $K \equiv [A - (\ell+1)k^{-\ell}]$

and $A = \frac{2(\ell+\frac{1}{2})}{k^\ell(\ell^2+\ell+1)}$, $k \equiv \frac{m}{2}$.

From the properties of Legendre polynomials, one finds the following relations

$$\int_{-1}^{1} P_\ell(x)\, dx = 0 \qquad \text{for all } \ell \geq 0 \tag{51}$$

$$\int_{-1}^{1} P_\ell^2(x)\, dx = \frac{2}{2\ell+1} \qquad \text{for all } \ell \geq 0 \tag{52}$$

and the recursion formula

$$(1-x^2)P_\ell'(x) = \ell P_{\ell-1}(x) - \ell x P_\ell(x), \tag{53}$$

hence

$$\int_{-1}^{1} P_\ell'^2(x)(1-x^2)dx = \int_{-1}^{1} P_\ell'(x)[\ell P_{\ell-1}(x) - \ell x P_\ell(x)]dx$$

$$= \int_{-1}^{1} \ell P_\ell'(x)P_{\ell-1}(x)dx - \int_{-1}^{1} \ell x P_\ell'(x)P_\ell(x)dx . \tag{54}$$

It can also easily be shown that

$$\int_{-1}^{1} P'_\ell(x) P_{\ell-1}(x)\, dx = 2 \qquad \text{for all} \quad \ell \geq 0$$

and

$$\int_{-1}^{1} x P'_\ell(x) P_\ell(x)\, dx = \frac{2\ell}{2\ell+1} \qquad \text{for all} \quad \ell \geq 0 .$$

Thus, (54) reduces to

$$\int_{-1}^{1} P_\ell^2(x)^2 (1-x^2) dx = 2\ell - \frac{\ell(2\ell)}{2\ell+1} = \frac{2\ell(\ell+1)}{(2\ell+1)} . \tag{55}$$

Having these relations at hand, the integral of (50) can easily be carried out by first changing the variable from θ to x with $x = \cos\theta$ and then applying the proper relations from (52), (54) and (55). The minimal area thus obtained is given by

$$A_{min} = 64k^2 + \frac{32\pi\varepsilon^2}{(2\ell+1)} \{[2A^2 - 8HA + 4KA + 3H] + A^2 \ell(\ell+1)\} + O(\varepsilon^3) + \ldots \tag{56}$$

So, explicitly, for $\ell \geq 1$

$$A_{min} = 16\pi m^2 - \frac{32\pi\varepsilon^2 (\ell-1)(\ell+2)}{(2\ell+1)k^{2\ell}(\ell^2+\ell+1)} + \ldots$$

$$= 16\pi\left(m^2 - \frac{2\varepsilon^2}{(\frac{m}{2})^{2\ell}} \left[\frac{(\ell-1)(\ell+2)}{(2\ell+1)(\ell^2+\ell+1)}\right] + \ldots\right) \tag{57}$$

and for $\ell = 0$

$$A_{min} = 16\pi m^2 + 128\pi k\varepsilon + 64\pi\varepsilon^2$$

$$= 16\pi(m + 2\varepsilon)^2 \ . \qquad (58)$$

To sum up results obtained so far: for a time-symmetric initial hypersurface with a slightly perturbed Schwarzschild 3-geometry,

$$ds_p^2 = (1 + \frac{m}{2r} + \frac{\varepsilon P_\ell}{r^{\ell+1}})^4 (dr^2 + r^2 d\Omega^2)$$

the minimal surface is a sphere with radius

$$r_{min} = \frac{m}{2} + \varepsilon \frac{2(\ell+\frac{1}{2})}{(\frac{m}{2})^{\ell}(\ell^2+\ell+1)} P_\ell(\cos\theta) + O(\varepsilon^2) \ ,$$

and the area of the minimal surface is, for $\ell \geq 1$

$$A_{min} = 16\pi \left\{ m^2 - \frac{2\varepsilon^2}{(\frac{m}{2})^{2\ell}} \left[\frac{(\ell-1)(\ell+2)}{(2\ell+1)(\ell^2+\ell+1)}\right] + \ldots \right\} \qquad (57)$$

and for the case $\ell = 0$

$$A_{min} = 16\pi(m + 2\varepsilon)^2 \ . \qquad (58)$$

Let us take a closer look at the last two equations. In equation (58), the minimal area looks like that of a Schwarzschild horizon with mass $m' = m+2\varepsilon$. This is hardly surprising. The perturbed metric for this case, $\ell = 0$, has a particularly simple form,

$$ds_p^2 = \{1 + \frac{m}{2} + \frac{\varepsilon}{r}\}^4 \ dr^2 + r^2 d\Omega^2\} \qquad (59)$$

in which the property $P_o(x) = 1$ has been used. The conformal factor can be rewritten as

$$\psi_p = 1 + \frac{m}{2r} + \frac{2\varepsilon}{2r} = 1 + \frac{m'}{2r} \tag{60}$$

where $m' \equiv m + 2\varepsilon$. In terms of m', the perturbed metric is

$$ds_p^2 = \{1 + \frac{m'}{2r}\}^4 \{dr^2 + r^2 d\Omega^2\} \quad . \tag{61}$$

This can easily be recognized to be the metric of a standard, unperturbed Schwarzschild geometry with mass m' measured from infinity. The event horizon of this geometry is the minimal sphere with radius

$$r_{min} = \frac{m'}{2} = \frac{m}{2} + \varepsilon$$

and with an area

$$A_{min} = 16\pi m'^2 = 16\pi(m + 2\varepsilon)^2 \quad . \tag{62}$$

A comparison with (58) reveals that it is actually the <u>exact</u> solution. The fact that an exact solution is reached when the perturbation is simple renders some credibility to this approximation method.

The perturbation depicted in equation (60) can be interpreted as lowering a spherical shell of mass 2ε quasi-statistically towards a static, spherical star or a Schwarzschild black hole. In the case where ε has a negative value, this is equivalent to extracting

a spherical shell quasi-statistically out to infinity. This of course can only be done before the star collapses through the horizon, that is before a black hole is formed. Alternatively, this can be viewed as adding a shell of negative mass to the black hole.

In the case where $\ell \geq 1$, the interpretation is less straightforward and clear-cut. The result (57) has the noteworthy feature that it does not have terms of order ε. In other words, the minimal area is independent of the sign of ε, but depends only on the magnitude of ε. In addition, the area is strictly less than the value $16\pi m^2$ irrespective of the choice of ε and ℓ (as long as $\ell \geq 1$). This feature, we shall show, indirectly confirms the Penrose conjecture of cosmic censorship. The conjecture essentially states that a singularity which arises from a gravitational collapse that starts off from a reasonable non-singular initial state can never be observed from infinity. The singularity, the conjecture goes, will always be clothed in an event horizon. The "big bang" singularity, in principle observable (Hawking 1967), does not belong to this class of singularities.

It has long been an established fact that stars depleted of nuclear fuel but still having more than a few solar-mass cannot find an equilibrium state. A massive cold star thus can have no other courses but

collapse to form a black hole. If the collapse is spherically symmetric, the end result must be a Schwarzschild black hole (or a Reissner-Nordstrøm black hole if the star has charge Q). Small deviation from spherical symmetry will not alter this final fate. The non-symmetric part will be radiated away in the form of gravitational waves(Price 1972). Part of the radiation will reach infinity, thus reducing the mass of the black hole, while the rest will be back-scattered across the horizon, thus increasing the mass and area of the horizon. It is, therefore, reasonable to expect the event horizon with radius

$$r = \frac{m}{2} + \varepsilon \frac{2(\ell+\frac{1}{2})}{(\frac{m}{2})^{\ell}(\ell^2+\ell+1)} P_\ell(\cos\theta)$$

of the perturbed metric

$$ds_p^2 = \{1 + \frac{m}{2r} + \frac{\varepsilon P_\ell}{r^{\ell+1}}\}^4 \{dr^2 + r^2 d\Omega^2\}$$

will eventually settle down to a Schwarzschild black hole with event horizon at

$$r = \frac{m_2}{2}$$

where m_2 is the final settled-down mass. Because of the radiation escaping to infinity, the final mass m_2 must be less than the initial mass m, i.e.

$$m_2 < m \quad . \tag{63}$$

The area of this final Schwarzschild event horizon is $16\pi m_2^2$.

After the perturbed horizon settles down to a new Schwarzschild horizon with mass m_2 and area $16\pi m_2^2$, the second law requires

$$16\pi m_2^2 > 16\pi\left\{m^2 - \frac{2\varepsilon^2}{(\frac{m}{2})^{2\ell}}\left[\frac{(\ell+1)(\ell+2)}{(2\ell+1)(\ell^2+\ell+1)}\right]\right\} \quad . \tag{64}$$

The amount of mass radiated away is $m - m_2$. Define the efficiency of radiation, α, by

$$\alpha \equiv \frac{m - m_2}{m} = 1 - \frac{m_2}{m} \quad . \tag{65}$$

Taking the square root of (64) gives

$$m_2 > m\left(1 - c\,\frac{\varepsilon^2}{m^2}\right)^{\frac{1}{2}} \tag{66}$$

where we have put

$$c \equiv \frac{2}{(\frac{m}{2})^{2\ell}}\left[\frac{(\ell-1)(\ell+2)}{(2\ell+1)(\ell^2+\ell+1)}\right] \quad .$$

With the help of (66), one can place an upper limit on α, i.e.

$$\alpha < 1 - \left(1 - c\,\frac{\varepsilon^2}{m^2}\right)^{\frac{1}{2}}$$

$$= 1 - \left[1 - \frac{1}{2}\,c\,\frac{\varepsilon^2}{m^2} - \frac{1}{8}\,c^2\,\frac{\varepsilon^4}{m^4} - \ldots\right] \quad .$$

Neglecting higher order terms, this reduces to

$$\alpha < \frac{1}{2} c \frac{\varepsilon^2}{m^2} \quad . \tag{67}$$

Thus we see that the radiation process cannot carry away more than $\frac{1}{2} c \frac{\varepsilon^2}{m^2}$ of its original mass.

The second law of black hole dynamics (Hawking 1972) states that in a spacetime containing no naked singularities, i.e. asymptotically predictable, the area of event horizon can only increase towards future. In hope of searching for a naked singularity, Penrose and Gibbons tried, but failed, to reach a contradiction with the second law. Their idea is essentially to construct time-symmetric initial data that has the same area of event horizon of a Schwarzschild solution but with less mass. To settle down to a Schwarzschild black hole, it either (1) has to radiate energy towards infinity and yet increases its mass, which is physically impossible, or (2) it has to decrease the area of the event horizon which contradicts the second law. In view of this and other results, Hawking (1973) conjectured that a contradiction of this type cannot be reached through Penrose's method.

As already pointed out, the area of (57) is strictly less than $16\pi m^2$ regardless of the choice of ε or ℓ ($\ell \geq 1$). Thus by this method of perturbation,

it is impossible to construct an event horizon, having an area larger than $16\pi m^2$ but with gravitational mass m. This result lends support to Hawking's conjecture.

IV. The Reissner-Nordstrom Initial Data

The spacetime manifold containing a charged, spherical body with mass m and charged Q is described by the Reissner-Nordstrom metric

$$ds^2 = (1 - \frac{2m}{r} + \frac{Q^2}{r^2})^{-1} dr^2 + r^2 d\Omega^2 - (1 - \frac{2m}{r} + \frac{Q^2}{r^2}) dt^2 . \qquad (68)$$

Since the metric coefficients satisfy the time-symmetry condition of (2.28), the RN geometry admits a time-symmetric hypersurface $\sum_i$. A Kruskal diagram similar to figure 4.1 can be constructed to exemplify this feature.

As in Schwarzschild geometry, the 3-geometry of the time-symmetric hypersurface $\sum_i$ is given by the spatial part of (68), i.e.

$$^{(3)}ds_R^2 = (1 - \frac{2m}{r} + \frac{Q^2}{r^2})^{-1} dr^2 + r^2 d\Omega^2 \quad . \qquad (69)$$

The RN metric, being a solution to the full Einstein field equation, must also satisfy the constraint equation

$$\frac{1}{2}\ {}^3R = -\kappa T_4{}^4 \quad .$$

In order to take advantage of the much simpler constraint

$$4\psi^{-5}\nabla^2\psi = -\kappa T_4{}^4 \;,$$

the 3-metric of (69) has first to be converted into an isotropic form, i.e.

$$^{(3)}ds_R^2 = \psi_R^4[d\bar{r}^2 + \bar{r}^2 d\Omega^2] \;. \tag{70}$$

Comparing coefficients of (69) and (70), we get

$$\psi_R^4 d\bar{r}^2 = [1 - \frac{2m}{r} + \frac{Q^2}{r^2}]^{-1} dr^2 \tag{71}$$

and

$$\psi_R^4 \bar{r}^2 = r^2 \;. \tag{72}$$

Eliminate ψ_R^4 from (71) and (72),

$$\frac{d\bar{r}}{\bar{r}} = \frac{dr}{r[1 - \frac{2m}{r} + \frac{Q^2}{r^2}]} \;.$$

This can be integrated to give

$$\ln(\bar{r}) + \text{const.} = \ln[(r-m) + \sqrt{r^2 - 2mr + Q^2}] \;.$$

To determine the constant, one looks at spatial infinity where m and Q can be neglected,

$$\ln(\bar{r}) + \text{const.} = \ln(2r) \;, \quad \text{for large } r;$$

or, $\quad \text{const.} = 2 \;.$

With this, the transformation between $\bar{r}$ and r is given by

$$\bar{r} = \frac{1}{2}(r-m) + \frac{1}{2}(r^2 - 2mr + Q^2)^{\frac{1}{2}} \;, \tag{73}$$

and the inverse transformation

$$r = \bar{r}\left[\left(1 + \frac{m}{2\bar{r}}\right)^2 - \frac{Q^2}{4\bar{r}^2}\right] \quad . \tag{74}$$

Substitute (74) into (72),

$$\psi_R = \left[\left(1 + \frac{m}{2\bar{r}}\right)^2 - \frac{Q^2}{4\bar{r}^2}\right]^{\frac{1}{2}} \quad . \tag{75}$$

Thus we arrive at the RN metric in isotropic coordinate,

$$^{(3)}ds_R^2 = \left[\left(1 + \frac{m}{2\bar{r}}\right)^2 - \frac{Q^2}{4\bar{r}^2}\right]^{\frac{1}{2}}\{d\bar{r}^2 + \bar{r}^2 d\Omega^2\} \ . \tag{76}$$

One can now read the conformal factor ψ_R directly from (76).

Unlike the Schwarzschild geometry, the space-time of RN metric is not vacuum but electro-vacuum. To calculate the non-zero energy momentum tensor $T_\mu^{\ \nu}$, one has first to investigate the time-symmetric electro-magnetic problem.

By time-symmetry, one essentially means that the physics is invariant under a time reversion. Consider a current of infinite length; a time inversion will reverse the direction of the current, hence reversing the magnetic field. The electric field, however, is unaffected. Therefore, intuitively, one would require that $\underline{B}$ field vanishes with only $\underline{E}$ field remaining for a time-symmetric EM problem.

In more precise terms, the Maxwell equations in covariant form are

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0 \tag{77}$$

and

$$\frac{1}{\sqrt{-g}}\, \partial_{\alpha}(\sqrt{-g}\, F^{\alpha\beta}) = J^{\beta} \tag{78}$$

where

$$F_{\alpha\beta} \equiv A_{\beta|\alpha} - A_{\alpha|\beta} = A_{\beta,\alpha} - A_{\alpha,\beta}$$

$$F^{\alpha\beta} = g^{\alpha\gamma} g^{\beta\delta} F_{\gamma\delta}$$

$$A_{\alpha} = (A_i, -\phi)$$

and $J^{\beta} = 0$ for electro-vacuum.

The time-symmetric EM problem is defined by

$$F^{*}_{\alpha\beta}(x^{\mu})\,dx^{\alpha}\wedge dx^{\beta} = F^{*}_{\alpha\beta}(\bar{x}^{\mu})\,d\bar{x}^{\alpha}\wedge d\bar{x}^{\beta} \tag{79}$$

where

$$x^{\mu} = (x^{i}, t)$$

$$\bar{x}^{\mu} = (x^{i}, -t)$$

and $F^{*}_{\alpha\beta}$ is the dual of $F^{\alpha\beta}$ defined by

$$F^{*}_{\alpha\beta} \equiv \frac{1}{2}\, \varepsilon_{\alpha\beta\mu\nu} F^{\mu\nu} \quad . \tag{80}$$

From (79), it follows that

$$F^{*}_{ij}(x^{\mu}) = F^{*}_{ij}(\bar{x}^{\mu}) \tag{81}$$

and

$$F^{*}_{4i}(x^{\mu}) = -F^{*}_{4i}(\bar{x}^{\mu}) \quad . \tag{82}$$

At t = 0, one therefore must have

$$F^{*}_{4i}\Big|_{t=0} = 0 \tag{83}$$

$$\frac{\partial F^{*}_{ij}}{\partial x^4}\Bigg|_{t=0} = 0 \tag{84}$$

Equation (84) comes from the fact that at t = 0, F^{*}_{ij} is independent of t and that at $t \neq 0$, F^{*}_{ij} is an even function of t.

Using the definition of the dual electromagnetic field tensor (80), one can easily show that (83) and (84) are equivalent to

$$F^{ij}\Big|_{t=0} = 0 \qquad \text{and} \qquad \frac{\partial F^{4j}}{\partial x^4}\Bigg|_{t=0} = 0\ . \tag{85}$$

But

$$F^{4i} = g^{44}g^{ij}F_{4j}$$

and

$$F^{ij} = g^{ik}g^{jh}F_{hk}\ ,$$

and since on the time-symmetric hypersurface, all metric components are time-independent, it then follows from (85) that

$$F_{4j,4}\Big|_{t=0} = 0 \quad \text{and} \quad F_{ij}\Big|_{t=0} = 0\ . \tag{86}$$

To simplify the picture, we further assume that the EM field on $\sum_i$ is purely electric, or equivalently

$$F_{ij,4}\Big|_{t=0} = 0 \; . \tag{87}$$

With (86) and (87), the EM field equations reduce to

$$\partial_i(\sqrt{-g}\; F^{4i}) = 0$$

or

$$\partial_i(\sqrt{{}^3g}\; g^{ij}\sqrt{g^{44}}\; F_{4j}) = 0 \; . \tag{88}$$

The energy momentum tensor of an EM field is given by

$$4\pi T_\mu{}^\nu = F_{\mu\alpha}F^{\nu\alpha} - \frac{1}{4}\,\delta_\mu{}^\nu F_{\alpha\beta}F^{\alpha\beta} \; .$$

If the EM field is time-symmetric with conformally flat metric, we have

$$\begin{aligned} 4\pi T_4{}^4 &= F_{4i}F^{4i} - \frac{1}{2}\,F_{4i}F^{4i} \\ &= \frac{1}{2}\,g^{44}g^{ij}[F_{4i}F_{4j}] \\ &= \frac{1}{2}\,(g_{44}g_{11})^{-1}\,[(\partial_r\phi)^2 + \frac{1}{r^2}(\partial_\theta\phi)^2 + \frac{1}{r^2\sin^2\theta}(\partial_\phi\phi)^2]. \end{aligned}$$

Thus the constraint equation becomes

$$4\nabla^2\psi = (\psi)(-g_{44})^{-1}[(\phi_{,r})^2 + \frac{1}{r^2}(\phi_{,\theta})^2 + \frac{1}{r^2\sin^2\theta}(\phi_{,\phi})^2] \; . \tag{89}$$

For the unperturbed RN case, equations (88) and (89) become

$$\partial_r[\psi_R^2 r^2\sqrt{-g^{44}}\ \partial_r(\phi_R)] = 0 \tag{90}$$

and

$$4\nabla^2\psi_R = \psi_R(-g^{44})[(\phi_{R,\tau})^2] \tag{91}$$

Now consider the slightly perturbed RN metric with ψ and ϕ given by

$$\psi = \psi_R + \varepsilon P_\ell \psi_1(r) \tag{92}$$

$$\phi = \phi_R + \varepsilon P_\ell \phi_1(r) \tag{93}$$

where P_ℓ is the Legendre polynomial, and ϕ_1, ψ_1 functions of r only.

The perturbed RN 3-metric is given by

$$g_{ij} = \begin{pmatrix} \psi^4 & 0 & \\ 0 & \psi^4 r^2 & 0 \\ 0 & 0 & \psi^4 r^2 \sin^2\theta \end{pmatrix} \tag{94}$$

In terms of ψ, ϕ, the field equation of (88) reads

$$\partial_r[\sqrt{^3g}\ g^{11}\sqrt{-g^{44}}\ \partial_r\theta] + \partial_\theta[\sqrt{^3g}\ g^{22}\sqrt{-g^{44}}\ \partial_\theta\phi] = 0\ . \tag{95}$$

From (92), (93), one gets

$$\psi_{,r} = \psi_{R,r} + \varepsilon P_\ell \psi_{1,r}\quad ;\quad \psi_{,\theta} = -\varepsilon P_\ell' \sin\theta\, \psi_1$$

$$\psi_{,rr} = \psi_{R,rr} + \varepsilon P_\ell \psi_{1,rr}\ ;\quad \psi_{,\theta\theta} = \varepsilon\psi_1[P_\ell''(1-x^2) - xP_\ell']$$

where $x = \cos\theta$, and similar equations for $\phi_{,r}$, $\phi_{,\theta}$, etc.

With these and equation (94), (95) can be reduced to

$$L(\phi_1)[P_\ell\psi_R + \varepsilon P_\ell^2\psi_1] - 2P_\ell\psi_1\cdot\frac{\psi_{R,r}\phi_{R,r}}{\psi_R} + 2P_\ell[\phi_{1,r}\psi_{R,r}+\psi_{1,r}\phi_{R,r}]$$

$$+ 2\varepsilon P_\ell^2\phi_{1,r}\psi_{1,r} + \frac{2}{r^2}\,\varepsilon P_\ell'^2\sin^2\theta\,\phi_1\psi_1 = 0 \qquad (96)$$

where $L(\phi_1)$ is given by

$$L(\phi_1) \equiv [\partial_r\partial_r + \frac{2}{r}\,\partial_r + \frac{1}{2}\,\frac{g^{44}{}_{,r}}{g^{44}}\,\partial_r - \frac{\ell(\ell+1)}{r^2}]\,\phi_1(r) \quad . \qquad (97)$$

Similarly, the constraint equation of (89) can be reduced to

$$P_\ell\,\mathcal{L}(\psi_1) = \frac{1}{4}\,\psi_R g_{44}^{-1}[\varepsilon P_\ell^2\phi_{1,r}^2 + 2P_\ell\phi_{R,r}\phi_{1,r} + \varepsilon P_\ell'^2\sin^2\theta\,\phi_1^2/r^2]$$

$$+ \frac{1}{4}\,P_\ell\psi_1 g_{44}^{-1}[\phi_{R,r}^2 + \varepsilon^2P_\ell^2\phi_{1,r}^2 + 2\varepsilon P_\ell\phi_{1,r}\phi_{R,r}$$

$$+ \varepsilon^2P_\ell'^2\sin^2\theta\,\frac{\phi_1^2}{r^2}] \qquad (98)$$

$$\text{where}\quad \mathcal{L}(\psi_1) \equiv [\partial_r\partial_r + \frac{2}{r}\,\partial_r - \frac{\ell(\ell+1)}{r^2}]\,\psi_1(r)\ . \qquad (99)$$

If we linearize the equations by dropping terms involving ε^2, we get, from (96),

$$L(\phi_1) = 2\psi_1\psi_R^{-2}\psi_{R,r}\phi_{R,r} - 2\psi_R^{-1}[\phi_{1,r}\psi_{R,r} + \psi_{1,r}\phi_{R,r}] \qquad (100)$$

and from (98),

$$\mathcal{L}(\psi_1) = \frac{1}{4}\,\psi_1 g_{44}^{-1}(\phi_{R,r})^2 + \frac{1}{2}\,\psi_R g_{44}^{-1}\cdot\phi_{R,r}\phi_{1,r} \qquad (101)$$

where the operators L and $\mathcal{L}$ are defined by (97) and (99) respectively.

In obtaining the above two equations, we made use of the fact that $g^{44} = -N^{-2}$ can be set arbitrarily without affecting the time evolution of the initial data. To simplify calculation, we set the $g^{44}_{(p)}$ of the perturbed RN field equal to the unperturbed g^{44}. For a vivid explanation of this, see, for example, Gravitation by M.T.W. (pp. 526-528).

Admittedly, equations (100) and (101) are difficult to solve analytically. Any further investigation will have to employ numerical methods.

CHAPTER 4

THE (t,ϕ) REVERSIBLE IVP AND SOME CLASSES OF PERTURBATION

I. Introduction

A rotating star with more than two solar-mass will undergo gravitational collapse after its nuclear fuel is depleted. During the collapse, gravitational radiation receding to infinity carries away energy and angular momentum. However, it is extremely unlikely that all the angular momentum can be radiated away in this manner. The final product of such a collapse, as one would expect, will then be a stationary rotating black hole which must be axisymmetric and (t,ϕ) reversible. Such a black hole must be a Kerr black hole according to the Israel-Carter conjecture which has now been almost completely established.

In this chapter, the IVP with both axisymmetry and (t,ϕ) reversal is studied. The IVP is formulated in a way that is different from, but parallel to, Ernst's (1968) formulation. The formulation sheds some light on the relation between the vacuum field equations and the constraint equations. However, this clarification is gained at the expense of complicating other calculations. The Kerr solution in this formulation is studied. Perturbation of Kerr solution is carried out to study the properties of the minimal surface.

II. The Stationary, Axisymmetric and (t,ϕ) Reversible IVP

A stationary spacetime with axisymmetry admits two one-parameter isometry groups, $\phi_t : V \to V$ and the cyclic $\phi_\phi : V \to V$ whose killing vectors can be expressed as

$$\xi^\alpha_t = \delta^\alpha_t$$

and

$$\xi^\alpha_\phi = \delta^\alpha_\phi$$

in properly chosen coordinates. It then follows that in this preferred coordinate, the components of the metric tensor is independent of coordinates t and ϕ, i.e.

$$g_{\alpha\beta,3} = 0 \qquad \text{and} \qquad g_{\alpha\beta,4} = 0$$

where we put $x^3 = \phi$, $x^4 = t$.

The line element can only be a function of the two spatial coordinates (x^1, x^2),

$$ds^2 = g_{\alpha\beta}(x^A)dx^\alpha dx^\beta \quad , \qquad A,B,\ldots = 1,2.$$

This can be further simplified by imposing the (t,ϕ) reversal condition. By (t,ϕ) reversal, it is meant that the spacetime is invariant under a simultaneous reversal of (x^3,x^4) to $(-x^3,-x^4)$. This condition essentially demands that the black hole be rotating and that the axis of symmetry be the axis of rotation.

The simultaneous reversal of (x^3, x^4) will then preserve the direction of rotation.

The line element is preserved under the reversal of (x^3, x^4), i.e.

$$g_{\alpha\beta}(x^A)\,dx^\alpha dx^\beta = g_{\alpha\beta}(x^A)\,d\bar{x}^\alpha d\bar{x}^\beta \tag{1}$$

where

$$x^\alpha = (x^1, x^2, \phi, t)$$

and

$$\bar{x}^\alpha = (x^1, x^2, -\phi, -t) \ .$$

This is equivalent to the condition that

$$g_{3A} = g_{4A} = 0 \ . \tag{2}$$

Written in full, the line element now looks

$$ds^2 = g_{AB}dx^A dx^B + g_{33}(dx^3)^2 + 2g_{43}dx^4 dx^3 + g_{44}(dx^4)^2 . \tag{3}$$

On the initial hypersurface $\sum_i : t = 0$, the 3-metric is

$$ds^2 = g_{AB}dx^A dx^B + g_{33}(dx^3)^2 \ . \tag{4}$$

The extrinsic curvature of $\sum_i$ can be calculated from

$$K_{ij} = -\frac{1}{2N}\left[N_{i|j} + N_{j|i} - g_{ij,4}\right]$$

where

$$N_i = g_{4i} \qquad \text{and} \qquad N = (-g^{44})^{-1/2} \ .$$

This gives

$$K_{AB} = K_{33} = 0 \quad \text{and} \quad K_{3A} = -\frac{1}{2N}\left[g_{33}\left(\frac{g_{43}}{g_{33}}\right)_{,A}\right] . \tag{5}$$

With (5), the initial constraint equations (1-16), (1-17) reduce to

$$\begin{aligned} \tfrac{1}{2}\,{}^3R + K_{3A}K^{3A} &= \kappa T_4{}^4 \\ K_i{}^j{}_{|j} &= -\kappa T_i{}^4 \end{aligned} \tag{6}$$

From now on, we shall concentrate on vacuum field, i.e. $T_\mu{}^\nu = 0$ everywhere.

In the following, we give the derivation of IVP in Ernst formulation, followed by a brief account of Ernst (1968) formulation of the vacuum field equation.

Following Weyl, we adopt for the stationary, axisymmetric, vacuum field the metric,

$$ds^2 = e^{2(\nu-\lambda)}(d\rho^2 + dz^2) + \rho^2 e^{-2\lambda} d\phi^2 - e^{2\lambda}(dt - \omega d\phi)^2 \tag{7}$$

where ν, λ, ω are functions of (ρ, z) only. The corresponding 3-metric is

$${}^3ds^2 = e^{-2\lambda}[e^{2\nu}(d\rho^2 + dz^2) + (\rho^2 - e^{4\lambda}\omega^2)d\phi^2] . \tag{8}$$

This implies $\sum_i$ is conformal to a 3-space $\sum_*$ with metric

$$ds_*^2 = e^{2\nu}(d\rho^2 + dz^2) + (\rho^2 - e^{4\lambda}\omega^2)d\phi^2 . \tag{9}$$

Denote the curvature scalar and the metric tensor of $\sum_*$ by R_* and g^*_{ij} respectively.

From (54)-(56) of Chapter 2, the curvature scalars of $\sum_i$ and $\sum_*$ are related by

$$^3R = e^2 \left[R_* - 4\Delta_2\lambda - 2\Delta_1\lambda\right] \tag{10}$$

where

$$\Delta_1\lambda \equiv g_*^{ij}\lambda_{,i}\lambda_{,j} = g_*^{11}(\lambda_\rho^2 + \lambda_z^2] \tag{11}$$

$$\Delta_2\lambda \equiv g_*^{ij}(\lambda_{,ij} - \lambda_{,k}\Gamma^k_{*ij})$$

$$= g_*^{11}(\lambda_{\rho\rho} + \lambda_{zz}) + \frac{1}{2}\, g_*^{11} g_*^{33}(\lambda_\rho g^*_{33,\rho} + \lambda_z g^*_{33,z}). \tag{12}$$

For a 3-manifold with orthogonal basis, i.e. $g_{ij} = 0$ for $i \neq j$, the scalar curvature is given by the simple equation (Eisenhart)

$$R = \sum_{ij} \frac{1}{g_{ii}} \frac{1}{g_{jj}} R_{ijji} \qquad (i \neq j) \tag{13}$$

and

$$R_{ijji} = (g_{ii}g_{jj})^{-\frac{1}{2}}\{\partial_i[g_{ii}^{-\frac{1}{2}} \cdot \partial_i(g_{jj}^{\frac{1}{2}})] + \partial_j[g_{jj}^{-\frac{1}{2}} \cdot \partial_j(g_{ii}^{\frac{1}{2}})]$$

$$+ g_{kk}^{-1} \cdot (g_{ii}^{\frac{1}{2}})_{,k} \cdot (g_{jj}^{\frac{1}{2}})_{,k}\} \quad \text{with } i,j,k \neq \tag{14}$$

With these equations, R_* is found to be

$$R_* = e^{-2\nu}\{2(\Delta\nu) + G^{-1}(\Delta G) - \frac{1}{2}\, G^{-2}(\nabla G)^2\} \tag{15}$$

where

$$G \equiv g^*_{33} = \rho^2 - e^{4\lambda}\omega^2 \quad ,$$

$$\Delta \equiv \partial_\rho\partial_\rho + \partial_z\partial_z \quad , \quad \nabla \equiv (\partial_\rho, \partial_z)$$

and

$$(\nabla G)^2 \equiv G_\rho^2 + G_z^2 \quad .$$

Following Ernst (1968), replace $e^{2\lambda}$ by f. From (10)-(12), we obtain the scalar curvature of $\sum_i$,

$$R = fe^{-2\nu}\{2(\Delta\nu) + G^{-1}\Delta G - \frac{1}{2} G^{-2}(\nabla G)^2 - 2[f^{-1}(\Delta f) - f^{-2}(\nabla f)^2]$$
$$- f^{-1}(\nabla f\nabla G) - \frac{1}{2} f^{-2}(\nabla f)^2\} \quad . \tag{16}$$

To obtain the constraint equations, we need the expression of $K_{3A}K^{3A}$ in terms of the metric tensor,

$$K_{3A}K^{3A} = \frac{1}{4}\cdot N^{-2}\cdot G^{-1}\cdot f^2 e^{-2\nu}\{4\omega^2(\nabla f)^2 + f^2(\nabla\omega)^2 + 4f\omega(\nabla f)(\nabla\omega)$$
$$+ f^2\omega^2 G^{-2}(\nabla G)^2 - 4f\omega^2 G^{-1}(\nabla f)(\nabla G) - 2f\omega G^{-1}(\nabla\omega\nabla G)\} \tag{17}$$

where

$$N^2 = -(g^{44})^{-1} = \rho^2 f G^{-1} \quad .$$

Combining (16), (17) gives the first constraint equation for a stationary, axisymmetric vacuum field,

$$2\nabla\nu + G^{-1}\Delta G - \frac{1}{2} G^2(\Delta G)^2 - 2[f^{-1}\Delta f - f^{-2}(\nabla f)^2] - f^{-1}(\nabla f\cdot\nabla G)$$
$$- \frac{1}{2} f^{-2}(\nabla f)^2 + \frac{1}{2\rho^2}\{4\omega^2(\nabla f)^2 + f^2(\nabla\omega)^2 - f^2\omega^2 G^{-2}(\nabla G)^2$$
$$+ 4f\omega\nabla f\cdot\nabla\omega - 4f\omega^2 G^{-2}\nabla f\cdot\nabla G - 2f\omega G^{-1}(\nabla\omega\cdot\nabla G)\} = 0. \tag{18}$$

The second constraint equation $K_i{}^j{}_{|j} = 0$ can be similarly

calculated. The calculation is straightforward but horrendously lengthy and messy. The effort, however, is paid off in the relatively simple form of the outcome,

$$2\rho^2\omega f^{-1}[f\nabla^2 f - (\nabla f)^2 + (\nabla\Phi)^2] + f(\rho^2 + f^2\omega^2)[\Delta^*\omega + 2f^{-1}(\nabla f\cdot\nabla\omega)]$$
$$= 0 \qquad (19)$$

where

$$\Delta^* \equiv \partial_\rho\partial_\rho - \frac{1}{\rho}\,\partial_\rho + \partial_z\partial_z$$

$$\nabla^2 \equiv \partial_\rho\partial_\rho + \frac{1}{\rho}\,\partial_\rho + \partial_z\partial_z \quad .$$

Unlike the time-symmetric, conformally flat $\sum_i$, in Chapter 2, the constraint equations (18), (19) are tediously long and almost unmanageable. This difficulty is essentially the motivation for the new formulation presented in the next section.

The Ernst (1968) formulation of stationary, axisymmetric, vacuum field runs briefly as follows.

The line element is the same as equation (7).

The vacuum field equation $R_{\mu\nu} = 0$ can be broken down to

$$\nu_\rho = \rho(\lambda_\rho^2 - \lambda_z^2) - \frac{1}{4}\,\rho^{-1}e^{4\lambda}(\omega_\rho^2 - \omega_z^2) \qquad (20)$$

$$\nu_z = 2\pi\lambda_\rho\lambda_z - \frac{1}{2}\,\rho^{-1}e^{4\lambda}\omega_\rho\omega_z \qquad (21)$$

$$\nabla^2\lambda = -\frac{1}{2}\rho^{-2}e^{4\lambda}(\nabla\omega)^2 \tag{22}$$

$$\Delta^*\omega = -4\nabla\lambda\cdot\nabla\omega \quad . \tag{23}$$

The last two equations (22), (23) are coupled, linear, second order differential equations, and serve as the integrability condition for ν. In the rare case where exact solutions of λ and ω are found, ν can be integrated easily from (20), (21). Thus, concentrate on (22), (23).

Replace $e^{2\lambda}$ by f. Equations (22), (23) become

$$f\nabla^2 f = (\nabla f)^2 - f^4\rho^{-2}(\nabla\omega)^2 \tag{24}$$

and

$$\frac{\partial}{\partial\rho}\left[\frac{f^2}{\rho}\frac{\partial\omega}{\partial\rho}\right] + \frac{\partial}{\partial z}\left[\frac{f^2}{\rho}\frac{\partial\omega}{\partial z}\right] = 0 \quad . \tag{25}$$

If we define

$$\frac{f^2}{\rho}\frac{\partial\omega}{\partial\rho} = P(\rho,z) \quad , \qquad \frac{f^2}{\rho}\frac{\partial\omega}{\partial z} = -Q(\rho,z) \quad , \tag{26}$$

equation (25) then becomes the condition for an exact differential, i.e. $\partial P/\partial\rho = \partial Q/\partial z$. This means that there exists a function $\Phi(\rho,z)$ whose total differential can be expressed as

$$d\Phi = P(\rho,z)dz + Q(\rho,z)d\rho \ .$$

The function Φ is uniquely defined by

$$\frac{\partial\Phi}{\partial z} = \frac{f^2}{\rho}\omega_\rho \quad , \qquad \frac{\partial\Phi}{\partial\rho} = -\frac{f^2}{\rho}\omega_z \quad . \tag{27}$$

With (27), the condition that $\omega_{\rho z} = \omega_{z\rho}$ can be rewritten as

$$\nabla\cdot(\frac{1}{f^2}\nabla\Phi) = 0$$

or

$$f\nabla^2\Phi - 2\nabla f\cdot\nabla\Phi = 0 \ . \tag{28}$$

The vacuum field problem for an axisymmetric, stationary, gravitational field then boils down to finding solutions for the coupled differential equations,

$$\left.\begin{aligned} &f\nabla^2 f = (\nabla f)^2 - (\nabla\Phi)^2 \\ \text{and} \quad & \\ &f\nabla^2\Phi = 2\nabla f\cdot\nabla\Phi \quad . \end{aligned}\right\} \tag{29}$$

On theoretical ground, a solution of the vacuum field equation $R_{\mu\nu} = 0$ is automatically a solution to the initial constraint equation since they are nothing but

$$G^4_\mu \equiv R^4_\mu - \frac{1}{2}\,\delta^4_\mu R = 0$$

and

$$R \equiv g^{\alpha\beta}R_{\alpha\beta} \ .$$

However, due to the complexity of the constraint equations of (18), (19), the inner relation between the two sets of equations is far from clear. In hope of finding a clearer picture, the author substituted the vacuum

field solution of (29) into the constraint equations, and found them to be compatible. However, any inner relations are completely clouded by the lengthy calculation. This situation is improved in the new formulation.

III. A New Formulation of the Axisymmetric and (t,ϕ) Reversible IVP

III-1) The new formulation and IVP

The Ernst formulation of the stationary, axisymmetric, vacuum field problem is characterized by two functions f and ω. The function f is the g_{44} component of the metric tensor and $f\omega$ forms the g_{43} component. Despite all its advantage, it is clear from the last section that it certainly is not the most desirable formulation for IVP. In the following, we present a new formulation based on the g_{33} and g_{43} of the metric tensor. The IVP equations are stunningly simple in this formulation. However, in gaining this simplicity, we pay the price of losing some of the advantages of Ernst formulation.

It is almost established that the eventual gravitational field of a rotating black hole is stationary and axisymmetric. To reach such a final state, nonstationary processes like gravitational radiation

are involved. Throughout this thesis, the perturbation analysis employed refers essentially to these non-stationary processes. In other words, we are interested in initial data that are only momentarily stationary and differ slightly from the final state. In this spirit, the IVP will be formulated with axisymmetry and (t,ϕ) reversal, with stationary condition assumed later on as a special case.

Similar to section II, an axisymmetric spacetime admits a Killing vector field $\xi^\alpha_\phi = \delta^\alpha_\phi$. The metric tensor in this coordinate is ϕ-independent, i.e. $g_{\alpha\beta,3} = 0$ where we put $x^3 = 0$. The line element is a function of (x^A,t) only, i.e.

$$ds^2 = g_{\alpha\beta}(x^A,t)dx^\alpha dx^\beta \ .$$

The (t,ϕ) reversal condition then requires

$$(1)\quad g_{AB}(x^A,t) = g_{AB}(x^A,-t); \qquad g_{33}(x^A,t) = g_{33}(x^A,-t)$$

$$g_{43}(x^A,t) = g_{43}(x^A,-t); \qquad g_{44}(x^A,t) = g_{44}(x^A,-t)$$

$$(2)\quad g_{3A}(x^A,t) = -g_{3A}(x^A,-t); \quad g_{4A}(x^A,t) = -g_{4A}(x^A,-t) \ .$$

On the initial spacelike hypersurface $\sum_i : t = 0$, these requirements become

$$(1)\quad g_{AB,4} = g_{33,4} = g_{43,4} = g_{44,4} = 0 \qquad \text{on } t = 0,$$

$$(2)\quad g_{3A} = g_{4A} = 0 \qquad \text{on } t = 0 \ .$$

Thus, at $t = 0$, the line element reduces to

$$ds^2 = g_{AB}(x^A)dx^A dx^B + g_{33}(x^A)(dx^3)^2 + 2g_{43}dx^4dx^3 + g_{44}(dx^4)^2. \tag{30}$$

Since the metric of any 2-space can be reduced to an isotropic form (Israel, 1972)*

$$^{(2)}ds^2 = f(x^A)[(dx^1)^2 + (dx^2)^2] \quad ,$$

the line element of (30) can be written as

$$ds^2 = e^{2\psi}(d\rho^2 + dz^2) + \ell d\phi^2 + 2md\phi dt - fdt^2 \tag{31}$$

where we put $g_{33} = \ell$, $g_{43} = m$, $g_{44} = f$, and $x^1 = \rho$, $x^2 = z$, $x^3 = \phi$, $x^4 = t$ to simplify the notation.

The line element of (31) is characterized by four functions, viz. ψ, ℓ, m, f. To further facilitate calculation, we define three new functions $\tilde{\nu}$, $\tilde{\omega}$, σ by

$$\psi = \tilde{\nu} - \frac{1}{2}\ln \ell \tag{32}$$

$$\tilde{\omega} \equiv m/\ell \tag{33}$$

and

$$(\sigma\rho)^2 = -\det\begin{vmatrix} g_{33} & g_{43} \\ g_{43} & g_{34} \end{vmatrix} = f\ell + m^2 \quad . \tag{34}$$

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* Lecture notes.

With this new notation, the metric becomes

$$ds^2 = \ell^{-1}e^{2\tilde{\nu}}(d\rho^2 + dz^2) + \ell d\phi^2 + 2\ell\tilde{\omega}d\phi dt - \ell^{-1}[(\sigma\rho)^2 - \ell^2\tilde{\omega}^2]dt^2 . \tag{35}$$

Notice that the number of functions remains the same, i.e. four, but they are changed to $\tilde{\nu}$, ℓ, $\tilde{\omega}, \sigma$ instead.

The corresponding 3-metric for $\sum_i$ is

$$ds^2 = \ell^{-1}e^{2\tilde{\nu}}(d\rho^2 + dz^2) + \ell d\phi^2 . \tag{36}$$

The extrinsic curvature from (5) is

$$K_{3A} = -\frac{1}{2N}\left(g_{33}\left(\frac{g_{43}}{g_{33}}\right)_{,A}\right)$$

where

$$N = (-g^{44})^{\frac{1}{2}} = \left[\frac{\ell}{(\sigma\rho)^2}\right]^{\frac{1}{2}} = \ell^{\frac{1}{2}}/(\sigma\rho) .$$

Hence

$$K_{31} = -\frac{\ell^{3/2}}{2(\sigma\rho)}\tilde{\omega}_{,1} \quad ; \quad K_{32} = -\frac{\ell^{3/2}}{\sigma\rho}\tilde{\omega}_{,2} . \tag{37}$$

The constraint equation $K_i{}^j{}_{|j} = 0$ can be reduced to

$$K_{31,1} + K_{32,2} + \frac{1}{2}K_{31}\frac{g_{33,1}}{g_{33}} + \frac{1}{2}K_{32}\frac{g_{33,2}}{g_{33}} = 0 . \tag{38}$$

Substituting (37) into the above equation, one arrives at

$$\left(\frac{\ell^{3/2}}{\sigma\rho}\tilde{\omega}_{,1}\right)_{,1} + \left(\frac{\ell^{3/2}}{\sigma\rho}\tilde{\omega}_{,2}\right)_{,2} + \frac{1}{2}\frac{\ell^{1/2}}{\sigma\rho}\tilde{\omega}_{,1}\ell_{,1}$$
$$+ \frac{1}{2}\frac{\ell^{1/2}}{\sigma\rho}\tilde{\omega}_{,2}\ell_{,2} = 0 .$$

Further simplification gives,

$$\frac{\partial}{\partial\rho}\left(\frac{\ell^2}{\sigma\rho}\,\tilde{\omega}_\rho\right) + \frac{\partial}{\partial z}\left(\frac{\ell^2}{\sigma\rho}\,\tilde{\omega}_z\right) = 0 \;. \tag{39}$$

This equation implies that there exists a function $\tilde{\Phi}$ whose total differential can be expressed as

$$d\tilde{\Phi} = -\left(\frac{\ell^2}{\sigma\rho}\,\tilde{\omega}_z\right)d\rho + \left(\frac{\ell^2}{\sigma\rho}\,\tilde{\omega}_\rho\right)dz \;.$$

The function $\tilde{\Phi}$ is uniquely defined by

$$\frac{\partial\tilde{\Phi}}{\partial\rho} = -\,\frac{\ell^2}{\sigma\rho}\,\tilde{\omega}_z \;, \qquad \frac{\partial\tilde{\Phi}}{\partial z} = \frac{\ell^2}{\sigma\rho}\,\tilde{\omega}_\rho \;. \tag{40}$$

With (40), the condition that $\tilde{\omega}_{\rho z} = \tilde{\omega}_{z\rho}$ can be rewritten as

$$\nabla\cdot\left(\frac{\sigma}{\ell^2}\,\nabla\tilde{\Phi}\right) = 0 \;. \tag{41}$$

Thus one can view the constraint equation $K_i{}^j{}_{|j} = 0$ as saying that there exists a function $\tilde{\Phi}$ which obeys (41).

An even more rewarding result appears when one tries to reduce the second constraint equation

$${}^3R + 2K_{3A}K^{3A} = 0 \;.$$

The last term can be expressed as

$$\begin{aligned} K_{3A}K^{3A} &= g^{33}g^{11}(K_{32}^2 + K_{32}^2) \\ &= \frac{1}{4\ell}\,e^{-2\tilde{\nu}}[\ell^4\rho^{-2}\tilde{\omega}_\rho^2 + \ell^4\rho^{-2}\tilde{\omega}_z^2) \\ &= \frac{1}{4\ell}\,e^{-2\nu}[\nabla\tilde{\Phi}]^2 \;. \end{aligned} \tag{42}$$

The curvature scalar ${}^3R$ can be calculated directly from (13) and (14). From (14), one gets

$$R_{1221} = \ell^{-1} e^{2\tilde{\nu}} \{ \Delta\tilde{\nu} + \frac{1}{2} (\frac{\nabla\ell}{\ell})^2 - \frac{1}{2} \Delta\ell \}$$

$$R_{1331} = \frac{1}{2} \ell_{,11} + \frac{1}{2} (-\ell_{,1}\tilde{\nu}_{,1} + \ell_{,2}\tilde{\nu}_{,2}) - \frac{1}{4} \frac{\ell_{,2}^2}{\ell}$$

$$R_{2332} = \frac{1}{2} \ell_{,22} + \frac{1}{2} (\ell_{,1}\tilde{\nu}_{,1} - \ell_{,2}\tilde{\nu}_{,2}) - \frac{1}{4} \frac{\ell_{,1}^2}{\ell} \quad .$$

And from (13),

$$ {}^3R = 2\ell \, e^{-\tilde{\nu}} \{ \Delta\tilde{\nu} + \frac{1}{4} (\frac{\nabla\ell}{\ell})^2 \} \quad . \tag{43}$$

Putting (42) and (43) together, one gets an astonishingly simple equation

$$\Delta\tilde{\nu} = - \frac{1}{4\ell^2} [(\nabla\ell)^2 + (\nabla\tilde{\Phi})^2] \ . \tag{44}$$

This new formulation can be summed up briefly as follows: At $t = 0$, the line element of an axi-symmetric, (t,ϕ) reversible spacetime is

$$ds^2 = e^{2\psi}[(dx^1)^2 + (dx^2)^2] + g_{33}d\phi^2 + 2g_{43}dtd\phi + g_{44}dt^2$$

Define four functions ℓ, $\tilde{\omega}$, $\tilde{\nu}$, σ by equations (32), (33), (34) such that the line element acquires the form,

$$ds^2 = \ell^{-1}e^{2\tilde{\nu}}(d\rho^2 + dz^2) + \ell d\phi^2 + 2\ell\tilde{\omega}d\phi dt - \ell^{-1}[(\sigma\rho)^2 - \ell^2\tilde{\omega}^2]dt^2.$$

With this, the constraint equations for the initial data on $\sum_i : t = 0$ reduce to

$$\nabla \cdot \left(\frac{\sigma}{\ell} \nabla \tilde{\Phi}\right) = 0$$

and

$$\Delta \tilde{\nu} = -\frac{1}{4\ell^2}\left[(\nabla \ell)^2 + (\nabla \tilde{\Phi})^2\right] .$$

III-2) The stationary vacuum field equation

In the following, we apply the new formulation to the axisymmetric, stationary vacuum field equation.

The line element can be written as

$$ds^2 = e^{2\psi}[(dx^1)^2 + (dx^2)^2] + \ell(dx^3)^2 + 2m dx^4 dx^3 - f(dx^4)^2. \tag{45}$$

The four unknown functions ψ, ℓ, m, f are to be determined by the vacuum field equation $R_{\mu\nu} = 0$. However, an ingenious observation by Weyl (1917) helped to reduce the number of unknown to three. This is done by first observing that the metric of (45) is form-invariant under a transformation of the type,

$$\bar{x}^1 + i\bar{x}^2 = F(x^1 + ix^2) \tag{46}$$

and secondly, that the vacuum field equation $R^3_3 + R^4_4 = 0$ can be reduced to

$$\partial_1\partial_1(D) + \partial_2\partial_2(D) = 0 , \tag{47}$$

where

$$D^2 \equiv -\det\begin{vmatrix} \ell & m \\ m & -f \end{vmatrix} . \tag{48}$$

Therefore D is a harmonic function. From elementary mathematics, one can easily find the conjugate harmonic function z such that

$$D + iz = F(x^1 + ix^2) . \tag{49}$$

Denote D by another name ρ. Using (49), one therefore can always transform the line element of (45) into

$$ds^2 = e^{2\psi}[d\rho^2 + dz^2] + \ell d\phi^2 + 2md\phi dt - fdt^2$$

where ℓ, m, f are no longer independent functions, but are related by (48), i.e.

$$\rho^2 = \ell f + m^2 \quad .$$

Similar to (32), (33), we define $\tilde{\nu}$ and $\tilde{\omega}$ by

$$\psi = \tilde{\nu} - \frac{1}{2} \ln \ell$$

$$\tilde{\omega} \equiv m/\ell$$

and from (48),

$$f = \frac{m^2 - \rho^2}{\ell} = \ell^{-1}(\ell^2\tilde{\omega}^2 - \rho^2) \; .$$

The line element now has the form

$$ds^2 = \ell^{-1}e^{2\tilde{\nu}}(d\rho^2 + dz^2) + \ell d\phi^2 + 2\ell\tilde{\omega}d\phi dt - \ell^{-1}[\rho^2 - \ell^2\tilde{\omega}^2]dt^2. \tag{50}$$

Comparing (35) and (50), one immediately notices that the two metrics differ by a function σ. If one let

$\sigma = 1$, equation (50) falls out from (35). In fact, σ can always be reduced to 1 for stationary cases (see appendix 1).

Digress to IVP for a moment. By putting $\sigma = 1$, we obtain the constraint equations for stationary, axisymmetric (t,ϕ) reversible, vacuum spacetime,

$$\nabla\cdot\left[\frac{1}{\ell^2}\,(\nabla\tilde{\Phi})\right] = 0 \tag{41a}$$

and

$$\Delta\tilde{\nu} = -\frac{1}{4\ell^2}\left[(\nabla\ell)^2 + (\nabla\tilde{\Phi})^2\right] \tag{44a}$$

where

$$\tilde{\Phi}_\rho \equiv -\frac{\ell^2}{\rho}\,\tilde{\omega}_z\,, \qquad \tilde{\Phi}_z = \frac{\ell^2}{\rho}\,\tilde{\omega}_\rho \quad .$$

In the coordinates (ρ, z, ϕ, t), the vacuum field equation $R^3_3 + R^4_4 = 0$ is always trivially satisfied. The rest of the vacuum field equations can be easily shown to reduce to (Khan, 1964)

$$\ell\Delta^*m = m\Delta^*\ell \qquad (\longleftrightarrow R^3_4 = 0) \tag{51}$$

$$f\Delta^*m = m\Delta^*f \qquad (\longleftrightarrow R^4_3 = 0) \tag{52}$$

$$f\Delta^*\ell = \ell\Delta^*f \qquad (\longleftrightarrow R^3_3 - R^4_4 = 0) \tag{53}$$

and

$$\psi_{,z} = -\frac{1}{4\rho}\left[\ell_{,\rho}f_{,z} + \ell_{,z}f_{,\rho} + 2m_{,\rho}m_{,z}\right] \tag{54}$$

$$\psi_{,\rho} = -\frac{1}{4\rho}\left[\ell_{,\rho}f_{,\rho} - \ell_{,z}f_{,z} + m^2_{,\rho} - m^2_{,z}\right] . \tag{55}$$

One should note that only two of the three equations (51), (52), (53) are independent.

In terms of functions $\tilde{\omega}$, $\tilde{\nu}$, equation (51) becomes

$$\ell\Delta^*\tilde{\omega} + 2\ell\nabla\tilde{\omega}\cdot\nabla\ell = 0 .$$

A closer observation shows that this is nothing but

$$\frac{\partial}{\partial\rho}\left(\frac{\ell^2}{\rho}\tilde{\omega}_\rho\right) + \frac{\partial}{\partial z}\left(\frac{\ell^2}{\rho}\tilde{\omega}_z\right) = 0 . \tag{56}$$

From section II, one should immediately recognize (56) implies that there exists a function $\tilde{\Phi}$ defined by

$$\tilde{\Phi}_\rho \equiv -\frac{\ell^2}{\rho}\tilde{\omega}_z , \qquad \tilde{\Phi}_z \equiv \frac{\ell^2}{\rho}\tilde{\omega}_\rho . \tag{57}$$

And the condition $\tilde{\omega}_{\rho z} = \tilde{\omega}_{z\rho}$ becomes

$$\nabla\cdot\left[\frac{1}{\ell^2}(\nabla\tilde{\Phi})\right] = 0 . \tag{58}$$

The trivially satisfied equation

$$\Delta^*(\rho^2) = 0$$

can be rewritten as

$$f\Delta^*\ell + \ell\Delta^*f + 2\nabla f\cdot\nabla\ell + 2m\Delta^*m + 2(\nabla m)^2 = 0$$

by replacing $\rho^2 = f\ell + m^2$. Using (53), (51) and (48), this can be simplified to

$$\rho^2\Delta^*\ell + \ell[\nabla\ell\cdot\nabla f + (\nabla m)^2] = 0 . \tag{59}$$

But

$$\nabla f = -\rho^2 \ell^{-2} \nabla \ell + \ell^{-1} \nabla(\rho^2) - \tilde{\omega}^2 \nabla \ell - 2\ell\tilde{\omega}\nabla\tilde{\omega}$$

and

$$(\nabla m)^2 = \tilde{\omega}^2 (\nabla \ell)^2 + \ell^2 (\nabla\tilde{\omega})^2 + 2\ell\tilde{\omega}\nabla\ell\cdot\nabla\tilde{\omega} \quad .$$

With the above two expressions substituted into (59), one gets

$$\ell\nabla^2 \ell - (\nabla\ell)^2 = -\frac{\ell^4}{\rho^2} (\nabla\tilde{\omega})^2 \ . \tag{60}$$

In terms of the function $\tilde{\Phi}$, this is just

$$\ell\nabla^2 \ell = (\nabla\ell)^2 - (\nabla\tilde{\Phi})^2 \ . \tag{61}$$

A straightforward calculation shows that (54), (55) can be re-expressed as

$$\tilde{\nu}_{,\rho} = \frac{\rho}{4\ell^2} [(\ell^2_{,\rho} - \ell^2_{,z}) - (\tilde{\Phi}^2_{,\rho} - \tilde{\Phi}^2_{,z})] \tag{62}$$

$$\tilde{\nu}_{,z} = \frac{\rho}{2\ell^2} [\ell_{,\rho}\ell_{,z} + \tilde{\Phi}_{,\rho}\tilde{\Phi}_{,z}] \ . \tag{63}$$

Rather to one's surprise, the vacuum field equations in this new formulation come out exactly the same as Ernst formulation with f replaced by ℓ. Thus, similar to Ernst formulation, the three unknown functions in the line element can be found by first solving the coupled equations

$$\nabla\cdot[\frac{1}{\ell^2} (\nabla\tilde{\Phi})] = 0 \tag{64}$$

and

$$\ell\nabla^2 \ell = (\nabla\ell)^2 - (\nabla\tilde{\Phi})^2 \ , \tag{65}$$

and then integrate (62), (63) to find $\tilde{\nu}$.

Because of the similarity between the two formulations, some of the desirable features are retained from Ernst formulation. For example, one can combine the two field equations (64), (65) by introducing a complex function $\tilde{\varepsilon}$,

$$\tilde{\varepsilon} \equiv \ell + i\tilde{\Phi} \quad . \tag{66}$$

The new field equation in terms of $\tilde{\varepsilon}$ is

$$(\mathrm{Re}\ \tilde{\varepsilon})\ \nabla^2\tilde{\varepsilon} = (\nabla\tilde{\varepsilon})^2 \ . \tag{67}$$

Furthermore, this field equation can be derived from a variational principle,

$$\delta \int \frac{(\nabla\tilde{\varepsilon})(\nabla\tilde{\varepsilon}^*)}{(\mathrm{Re}\ \tilde{\varepsilon})^2}\, dV = 0$$

where dV is the volume element of Euclidean 3-space.

III-3) A comparison of the two formulations

A quick look at the two formulations shows that the two sets of vacuum field equations are identical in form. If one replaces f by ℓ and Φ by $\tilde{\Phi}$, one goes from the Ernst formulation to the other. It therefore seems quite incongruous that the two sets of constraint equations should differ by such a great extent. However, a careful examination shows otherwise.

If one keeps both f, ℓ and m in the vacuum field equations $R_\mu{}^\nu = 0$, the symmetry between f and ℓ becomes apparent. The six vacuum field equations are

$$\ell \Delta^* m = m \Delta^* \ell$$

$$f \Delta^* m = m \Delta^* f$$

$$\ell \Delta^* f = f \Delta^* \ell$$

$$\Delta^* [f\ell + m^2] = 0$$

$$\psi_{,1} = -\frac{1}{4\rho} [\ell_{,1} f_{,1} - \ell_{,2} f_{,2} + (m_{,1})^2 - (m_{,2})^2]$$

and

$$\psi_{,2} = -\frac{1}{4\rho} [\ell_{,1} f_{,2} + \ell_{,2} f_{,1} + 2m_{,1}\, m_{,2}] \; .$$

An interchange of f and ℓ does not affect these equations. Thus one can expect the vacuum field equations would be identical in both formulations.

In the initial value problem, such a symmetry is manifestly lacking. The constraint equation bases itself on spacelike hypersurfaces instead of the complete spacetime manifold. The g_{44} component of the metric tensor enters the IVP in the form of the relatively insignificant lapse function $N = (-g^{44})^{-\frac{1}{2}}$. Therefore, the simpler the 3-metric, in particular, the g_{33} component, the simpler the constraint equations. In the $(\ell, \tilde{\omega})$ formulation, we essentially compress the complexity of g_{33} in Ernst formulation into two functions $\tilde{\nu}$ and ℓ, and hence greatly simplify the constraint

equations. However, this simplification is gained over the complication of the g_{44} component which turns out to be detrimental in finding all the known solutions to the Einstein field equation.

IV. Solutions in the $(\ell,\tilde{\omega})$ Formulation

To date, the only known axisymmetric, stationary solution to the vacuum field equation corresponding to the vacuum exterior field of a rotating black hole is the Kerr solution. It is commonly believed that this is probably the only solution. Thus, it is a great disadvantage of the $(\ell,\tilde{\omega})$ formulation that the Kerr solution cannot be deduced in a simple, straightforward manner as the Ernst formulation was capable of doing. By hook and by crook, the expression of $\tilde{\Phi}$ for the Kerr solution in $(\ell,\tilde{\omega})$ formulation is nevertheless obtained.

It was shown by introducing the complex function

$$\tilde{\varepsilon} \equiv \ell + i\tilde{\Phi} \quad , \tag{66}$$

the field equation acquires the form

$$(\mathrm{Re}\ \tilde{\varepsilon})\ \nabla^2\tilde{\varepsilon} = (\nabla\tilde{\varepsilon})^2 \ . \tag{67}$$

Following Ernst, define a function $\tilde{\xi}$ by

$$\tilde{\varepsilon} = \frac{\tilde{\xi}-1}{\tilde{\xi}+1} \quad . \tag{68}$$

A straightforward calculation gives

$$(\mathrm{Re}\ \tilde{\varepsilon}) = \frac{\tilde{\xi}\tilde{\xi}^* - 1}{(\tilde{\xi}+1)(\tilde{\xi}^*-1)}$$

$$\nabla^2\tilde{\varepsilon} = \frac{\ell\nabla^2\tilde{\xi}}{(\tilde{\xi}+1)^2} - \frac{4(\nabla\tilde{\xi})^2}{(\tilde{\xi}+1)^3}$$

and

$$(\nabla\tilde{\varepsilon})^2 = \frac{4(\nabla\tilde{\xi})^2}{(\tilde{\xi}+1)^4} \quad .$$

With these, the field equation (67) becomes

$$(\tilde{\xi}\tilde{\xi}^*-1)\nabla^2\tilde{\xi} = 2\tilde{\xi}^*(\nabla\tilde{\xi})^2 \quad . \qquad (69)$$

In prolate spheroidal coordinates (λ,θ,ϕ), the Laplacian operator ∇^2 is given by

$$\nabla^2 \equiv \frac{1}{b^2(\mathrm{sh}^2\lambda + \mathrm{sn}^2\theta)}\ \theta\{\frac{1}{\mathrm{sh}\ \lambda}\ \partial_\lambda(\mathrm{sh}\ \lambda\, \partial_\lambda) + \frac{1}{\mathrm{sh}\ \theta}\ \partial_\theta(\mathrm{sn}\ \theta\, \partial_\theta)\}$$

$$+ \frac{1}{b^2\mathrm{sh}^2\lambda\ \mathrm{sn}^2\theta}\ \partial_\phi^2$$

and the gradient operator ∇ is given by

$$\nabla \equiv \frac{1}{b(\mathrm{sh}^2\lambda+\mathrm{sn}^2\theta)^{\frac{1}{2}}}\ \partial_\lambda + \frac{1}{b(\mathrm{sh}^2\lambda+\mathrm{sn}^2\theta)^{\frac{1}{2}}}\ \partial_\theta + \frac{1}{b\ \mathrm{sh}\,\lambda\ \mathrm{sn}\,\theta}\ \partial_\phi$$

where $\mathrm{sh}\ \lambda \equiv \sinh\lambda$, $\quad \mathrm{sn}\ \theta \equiv \sin\theta$,

similarly $\mathrm{ch}\ \lambda \equiv \cosh\lambda$, $\quad \mathrm{cn}\ \theta \equiv \cos\theta$.

The transformation between the cylindrical coordinates (ρ,z,ϕ) and prolate spheroidal coordinates (λ,θ,ϕ) is

$$\rho = b \sinh\lambda \sin\theta, \quad z = b\cosh\lambda\cos\theta \quad \phi=\phi \, . \tag{70}$$

Since $\tilde{\xi}$ is independent of ϕ, equation (69) becomes

$$(\tilde{\xi}\tilde{\xi}^*-1)\{\frac{1}{\text{sh }\lambda}\,\partial_\lambda[\text{sh }\lambda\,\partial_\lambda\tilde{\xi}] + \frac{1}{\text{sh }\theta}\,\partial_\theta[\text{sn }\theta\,\partial_\theta\tilde{\xi}\,]\}$$

$$= 2\tilde{\xi}^*[(\partial_\lambda\tilde{\xi})^2 + (\partial_\theta\tilde{\xi})^2] \, . \tag{71}$$

It is easy to verify that

(1) if F is a solution of (69), then $e^{iA}F$ is also a solution, where A is a real constant,

(2) that $\cosh\lambda$, $\cos\theta$, hence $\pm i\cosh\lambda$, $\pm i\cos\theta$ are solutions of (71),

(3) even though (71) is non-linear, $\cos\alpha\cosh\lambda \pm i\sin\alpha\cos\theta$, $\sin\alpha\cos\theta \pm i\cos\alpha\cosh\lambda$ are also solutions.

However, none of these solutions are asymptotically flat, hence have little physical meaning.

In the following, we obtain, by hook and by crook, the function $\tilde{\Phi}$ for the Kerr solution. The Kerr metric in Lindquist-Boyer coordinate (r,θ,ϕ) is given by

$$ds^2 = (r^2 + a^2\cos^2\theta)[\frac{dr^2}{r^2-2mr+a^2} + d\theta^2]$$

$$+ (r^2+a^2 + \frac{2mra^2\sin^2\theta}{r^2+a^2\cos\theta})\sin^2\theta\, d\phi^2$$

$$+ \frac{4mar\sin^2\theta}{r^2+a^2\cos^2\theta}\, d\phi dt - (1 - \frac{2mr}{r^2+a^2\cos^2\theta})\, dt^2 \, . \tag{72}$$

The metric of an axisymmetric, stationary spacetime is given by equation (50), i.e.

$$ds^2 = \ell^{-1} e^{2\tilde{\nu}}(d\rho^2 + dz^2) + \ell d\phi^2 + 2\ell\tilde{\omega} d\phi dt - f dt^2 \qquad (73)$$

where we have $f\ell + \ell^2\tilde{\omega}^2 = \rho^2$.

Since the Kerr solution is a solution to the vacuum field equation irrespective of the formulation, one gets, by comparing (72) and (73),

$$\ell_{Kerr} = (r^2 + a^2 + \frac{2mra^2\sin^2\theta}{r^2+a^2\cos^2\theta}) \sin^2\theta \qquad (74)$$

$$\ell_K\tilde{\omega}_K = \frac{2mar \sin^2\theta}{r^2+a^2\cos^2\theta} \quad . \qquad (75)$$

The Lindquist-Boyer coordinates (r,θ,ϕ) are connected to the "cylindrical" coordinates (ρ,z,ϕ) by

$$\left.\begin{aligned} \rho &= (r^2 - 2mr + a^2)^{\frac{1}{2}} \sin\theta \\ z &= (r - m) \cos\theta \ . \end{aligned}\right\} \qquad (76)$$

Therefore,

$$d\rho^2 + dz^2 = [(r-m)^2 + (m^2-a^2)\cos^2\theta]\{\frac{dr^2}{r^2-2mr+a^2} + d\theta^2\}. \quad (77)$$

Again comparing coefficient,

$$\ell^{-1}e^{2\tilde{\nu}}[(r-m)^2 - (m^2-a^2)\cos^2\theta] = (r^2+a^2\cos^2\theta)$$

or

$$e^{2\tilde{\nu}} = \ell(r^2+a^2\cos^2\theta)/[(r-m)^2-(m^2-a^2)\cos^2\theta]. \quad (78)$$

From (57)

$$\left.\begin{aligned}\tilde{\Phi}_\rho &\equiv -\frac{\ell^2}{\rho}\,\tilde{\omega}_z = -\frac{\ell^2}{\rho}\left(\frac{m}{\ell}\right)_z = -\frac{1}{\rho}\,(\ell m_z - m\ell_z)\\ \tilde{\Phi}_z &\equiv \frac{\ell^2}{\rho}\,\tilde{\omega}_\rho = -\frac{\ell^2}{\rho}\left(\frac{m}{\ell}\right)_\rho = +\frac{1}{\rho}\,(\ell m_\rho - m\ell_\rho)\end{aligned}\right\} \qquad (79)$$

where $m \equiv \ell\tilde{\omega}$. By chain-rule of differentiation, one gets

$$\left.\begin{aligned}\tilde{\Phi}_r r_\rho + \tilde{\Phi}_\theta \theta_\rho &= -\frac{1}{\rho}\,[\ell(m_r r_z + m_\theta \theta_z) - m(\ell_r r_z + \ell_\theta \theta_z)]\\ \tilde{\Phi}_r r_z + \tilde{\Phi}_\theta \theta_z &= \frac{1}{\rho}\,[\ell(m_r r_\ell + m_\theta \theta_\rho) - m(\ell_r r_\rho + \ell_\theta \theta_\rho)]\end{aligned}\right] \qquad (80)$$

where $r_\rho \equiv \partial r/\partial\rho$, etc. By noting that

$$\begin{vmatrix} \rho_r & \rho_\theta \\ z_r & z_\theta \end{vmatrix} \cdot \begin{vmatrix} r_\rho & r_z \\ \theta_\rho & \theta_z \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

one can solve to get

$$\left.\begin{aligned} r_\rho &= \frac{\rho(r-m)}{(\rho^2+z^2)}\,, & r_z &= \frac{(r^2-2mr+a^2)\cos\theta}{(\rho^2+z^2)}\\ \theta_\rho &= \frac{(r^2-2mr+a^2)^{\frac{1}{2}}\cos\theta}{\rho^2+z^2}\,, & \theta_z &= \frac{-(r-m)\sin\theta}{\rho^2+z^2}\end{aligned}\right\} \qquad (81)$$

These can be used to further simplify (80) to

$$\tilde{\Phi}_r = \frac{1}{\rho} \; \frac{\ell m_\theta - m\ell_\theta}{(r^2-2mr+a^2)^{\frac{1}{2}}}$$
$$\tilde{\Phi}_\theta = -\frac{1}{\rho}(r^2-2mr+a^2)^{\frac{1}{2}}(\ell m_r - m\ell_r) \tag{82}$$

By substituting (74) and (75) into (82), one can show, after a lengthy calculation,

$$\tilde{\Phi}_{r\theta} = \tilde{\Phi}_{\theta r} \; .$$

This assures that $\tilde{\Phi}$ can be integrated from (82) by

$$\tilde{\Phi} = \int \tilde{\Phi}_r \, dr + G(\theta)$$

where $G(\theta)$ is a function of θ only. With the help of (74), (75), one finds

$$\tilde{\Phi}_r = \frac{4mra^3 \sin^4\theta \cos\theta}{(r^2+a^2\cos^2\theta)^2} \quad . \tag{83}$$

A simple integration then gives

$$\tilde{\Phi} = -\frac{2ma^3 \sin^4\theta \cos\theta}{(r^2a^2\cos^2\theta)} + G(\theta) \; . \tag{84}$$

To determine $G(\theta)$: differentiate the expression for $\tilde{\Phi}$ in (84) with respect to θ; equate this result with the $\tilde{\Phi}_\theta$ obtained directly from (82); this then yields an expression for $dG(\theta)/d\theta$, i.e.

$$\frac{dG(\theta)}{d\theta} = 6\,ma\,\sin^3\theta \quad .$$

Hence

$$G(\theta) = \int 6ma \sin^3\theta \, d\theta = -6 \, ma \cos\theta (1 - \tfrac{1}{3} \cos^2\theta) + C.$$

Therefore, for the Kerr solution, we have

$$\tilde{\Phi}_{Kerr} = - \frac{2ma^3 \sin^4\theta \cos\theta}{(r^2+a^2\cos^2\theta)} - 6ma\cos\theta(1 - \tfrac{1}{3}\cos^2\theta) + C \tag{85}$$

where C is a constant.

V. Perturbation of Kerr Metric

V-1) In the $(\ell, \tilde{\omega})$ formulation of the axisymmetric stationary gravitational field, the initial constraint equations were shown to be

$$\Delta\tilde{\nu} = - \frac{1}{4\ell^2} [(\nabla\ell)^2 + (\nabla\tilde{\Phi})^2] \tag{44a}$$

and

$$\ell\Delta^{*2}\tilde{\omega} = - 2\nabla\ell\cdot\nabla\tilde{\omega} \quad . \tag{41a}$$

The second constraint equation essentially guarantees the existence of the function $\tilde{\Phi}$. Hence concentrate on the first constraint equation.

Consider a slightly perturbed Kerr geometry with $\tilde{\nu}_p$ such that it differs from the unperturbed Kerr $\tilde{\nu}$ by an infinitesimal amount, i.e.

$$\tilde{\nu}_p = \tilde{\nu} + \varepsilon\eta \tag{86}$$

where ε is an infinitesimal quantity and η a function

of (ρ,z) only. The constraint equation then demands

$$\Delta\eta = 0 \tag{87}$$

or

$$\frac{\partial^2\eta}{\partial\rho^2} + \frac{\partial^2\eta}{\partial z^2} = 0 \ . \tag{88}$$

Define a polar coordinate (μ,ϕ) on the 2-surface (ρ,z) by

$$\left.\begin{aligned} &\mu^2 = \rho^2 + z^2 \\ &\text{and} \\ &\rho = \mu \sin\phi \ , \quad z = \mu\cos\phi \end{aligned}\right\} \tag{89}$$

Since $\rho = (r^2-2mr+a^2)^{\frac{1}{2}}\sin\theta$, $z = (r-m)\cos\theta$, we have

$$\left.\begin{aligned} &\mu^2 = [(r^2-m)^2 + (a^2-m^2)\sin^2\theta] \\ &\text{and} \\ &\cos\phi = \frac{z}{\mu} = \frac{(r-m)}{\mu}\cos\theta \ . \end{aligned}\right\} \tag{90}$$

In (μ,ϕ) coordinates, equation (88) becomes

$$\eta_{\mu\mu} + \frac{1}{\mu^2}\eta_{\phi\phi} + \frac{1}{\mu}\eta_\mu = 0$$

where the subscript denotes partial differentiation. Using separation of variables, one can easily find a solution

$$\eta = \frac{\cos n\phi}{\mu^n} \tag{91}$$

where n is positive integer.

The 3-metric for the perturbed Kerr geometry is

$$ {}^3ds^2 = \ell^{-1} e^{2(\tilde{\nu}+\varepsilon\eta)} (d\rho^2 + dz^2) + \ell d\phi^2 \ , $$

where $\ell, \tilde{\nu}$ are given by equations (74) and (78) respectively, or

$$ {}^3ds^2 = e^{2\varepsilon\eta} (r^2+a^2\cos^2\theta)\{\frac{dr^2}{r^2-2mr+a^2} + d\theta^2\} $$

$$ + (r^2+a^2+\frac{2mra^2\sin^2\theta}{r^2+a^2\cos^2\theta}) \sin^2\theta \, d\phi^2 \ . \qquad (92) $$

The event horizon in the initial surface of (t,ϕ) reversal remains point-wise fixed under the isometry $(t,\phi) \longrightarrow (-t,-\phi)$. By the lemma in Chapter 3, the event horizon is then a geodesically complete submanifold, and hence an extremal 2-surface. As the event horizon is necessarily the surface with the minimum area, it then follows that the event horizon is the minimal surface in the (t,ϕ) reversible initial hypersurface. Therefore, finding the minimal surface is equivalent to finding the event horizon.

In the stationary case, the marginally trapped surface, which is both null and stationary, coincides with the event horizon. The event horizon is also the first horizon an infalling space-traveller encounters. The traveller is lost forever to an observer at infinity once he crosses this horizon. Combining

these two properties, the event horizon can be alternatively, but equivalently, defined as the outermost stationary null hypersurface.

Let the equation for a hypersurface be

$$f(x^{\alpha}) = 0 \; .$$

If the hypersurface is stationary, $f(x^{\alpha})$ must be independent of x^4; if it is a trapped surface in a geometry with axisymmetry, $f(x^{\alpha})$ is independent of $x^3 = \phi$; if it is null, $f(x^{\alpha})$ must have

$$g^{\alpha\beta} f_{,\alpha} f_{,\beta} = 0 \; .$$

Thus the event horizon satisfies

$$g^{11}(f_{,1})^2 + g^{22}(f_{,2})^2 = 0 \; . \tag{93}$$

For the perturbed geometry, this is just

$$e^{-2\varepsilon\eta} \frac{(r^2-2mr+a^2)}{r^2+a^2\cos^2\theta} f^2_{,r} + e^{-2\varepsilon\eta} . \frac{1}{r^2+a^2\cos^2\theta} f^2_{,\theta} = 0 \; ,$$

or equivalently,

$$(r^2-2mr+a^2) f^2_{,r} + f^2_{,\theta} = 0 \; . \tag{94}$$

This equation, surprisingly enough, is independent of the perturbation function η, and is identical with the equation of the unperturbed Kerr event horizon. That is $f(x^A) = r - r_{\pm}$ are solutions to equation (94), and

that $r=r_+$ represents the outermost stationary null hypersurface. The quantities $r_\pm$ are solutions to $r^2-2mr+a^2 = 0$, i.e.

$$r_\pm = m \pm \sqrt{m^2 - a^2} \quad . \tag{95}$$

In contrast to the linearized solution (4-42a) for the perturbed Schwarzschild geometry, the equation for the minimal surface in the perturbed Kerr geometry,

$$r = m + \sqrt{m^2 - a^2}$$

is exact, and is independent of the perturbation η.

With this result, one is tempted into concluding that the perturbation does not affect the event horizon. However, this is not true as the radius r acquires different meanings in the different geometries. The metric contains information of both the geometry and the meaning of coordinates. As the metric changes, the meaning of the coordinates also varies. This can be best understood with the following consideration. In the Kerr geometry, the hypersurface r = constant represents a closed 2-surface with intrinsic 2-metric:

$${}^2ds^2_{Kerr} = (r^2+a^2\cos^2\theta)d\theta^2 + \ell d\phi^2 \tag{96}$$

and has an area of

$$A_{Kerr} = \int_0^{2\pi}\int_0^{\pi} (r^2+a^2\cos^2\theta)^{\frac{1}{2}}\ell^{\frac{1}{2}}d\theta d\phi \ . \tag{97}$$

For the perturbed geometry, r = constant also represents a closed 2-surface but with a <u>different</u> 2-metric

$$ {}^2ds_p^2 = e^{2\varepsilon\eta}(r^2+a^2\cos^2\theta)d\theta^2 + \ell d\phi^2 \tag{98} $$

and has a different area

$$ A_p = \int_0^{2\pi}\int_0^{\pi} e^{\varepsilon\eta}(r+a^2\cos^2\theta)^{\frac{1}{2}}\ell^{\frac{1}{2}}d\theta d\phi \quad . \tag{99} $$

One thus sees that the event horizon of (95) is actually different from the Kerr event horizon. To see this difference, we calculate the area from (97) and (99) for $r = m + \sqrt{m^2-a^2} \equiv r_+$. The area of a Kerr event horizon is

$$ A_K = \int_0^{2\pi}\int_0^{\pi} (r_+^2+a^2\cos^2\theta)^{\frac{1}{2}}(r_+^2+a^2+\frac{2mr_+^2+a^2\sin^2\theta}{r_+^2+a^2\cos^2\theta})^{\frac{1}{2}} \sin\theta\, d\theta\, d\phi $$

$$ = -2\pi\int_0^{\pi} 2m(m+\sqrt{m^2-a^2})\ d(\cos\theta) $$

$$ = 8\pi m(m+\sqrt{m^2-a^2}) \quad . \tag{100} $$

The area of the perturbed horizon is

$$ A_p = \int_0^{2\pi}\int_0^{\pi} e^{\varepsilon\eta}(r_+^2+a^2\cos^2\theta)^{\frac{1}{2}}(r_+^2+a^2+\frac{2mr_+a^2\sin^2\theta}{r_+^2+a^2\sin^2\theta})\sin\theta\, d\theta d\phi \tag{101} $$

with η evaluated at r_+, i.e.

$$ \eta\Big|_{r+} = \frac{\cos n\phi}{\mu^n}\Big|_{r+} = \frac{1}{(m^2-a^2)^{n/2}\cos^n\theta} \quad . $$

To ensure that $e^{\varepsilon\eta}$ remains finite when $\theta = \pi/2$, choose $n = 2\ell$ and

$$\eta\Big|_{r+} = \frac{-1}{(m^2-a^2)^{\ell}\cos^{2\ell}\theta} .$$

With this choice, $e^{\varepsilon\eta} \to 0$ at $\theta \to \pi/2$.

Denote $A = 1/(m^2-a^2)^{\ell}$ and $x = \cos\theta$, equation (101) now becomes

$$A_p = 4\pi m r_+ \int_{-1}^{1} e^{-\frac{\varepsilon}{A} x^{-2\ell}} dx . \tag{102}$$

To evaluate this integral, we perform a change of variable. Let $t = \alpha x^{-2\ell}$ where $\alpha \equiv \varepsilon/(m^2-a^2)^{\ell}$, then

$$x = \alpha^{\frac{1}{2\ell}} t^{-\frac{1}{2\ell}}$$

and

$$dx = \alpha^{\frac{1}{2\ell}} \left(-\frac{1}{2\ell}\right) t^{-\left(\frac{1}{2\ell}+1\right)} dt .$$

The integral now has the form

$$\begin{aligned}
A_p &= 8\pi m r_+ \int_0^1 e^{-\alpha x^{-2\ell}} dx \\
&= -8\pi m r_+ \int_{\alpha}^{\infty} e^{-t} \cdot \alpha^{\frac{1}{2\ell}} d\left(t^{-\frac{1}{2\ell}}\right) \\
&= -8\pi m r_+ \alpha^{\frac{1}{2\ell}} \left(e^{-\alpha} t^{-\frac{1}{2\ell}}\Big|_{\alpha}^{\infty} + \int_{\alpha}^{\infty} t^{-\frac{1}{2\ell}} e^{-t} dt \right) \\
&= 8\pi m r_+ \left(e^{-\alpha} - \alpha^{\frac{1}{2\ell}} \int_0^{\infty} e^{-} t^{-\frac{1}{2\ell}} dt + \alpha^{\frac{1}{2\ell}} \int_0^{\alpha} e^{-t} t^{-\frac{1}{2\ell}} dt \right) .
\end{aligned} \tag{103}$$

The first integral is just the gamma function $\Gamma(1-\frac{1}{2\ell})$ and the second integral can be expressed as

$$\alpha^{\frac{1}{2\ell}} \int_0^{\alpha} t^{-\frac{1}{2\ell}} \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \ldots + \frac{(-1)^n t^n}{n!} + \ldots\right) dt$$

$$= \alpha^{\frac{1}{2\ell}} \int_0^{\alpha} t^{-\frac{1}{2\ell}} \left(\sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!}\right) dt$$

$$= \sum_{n=0}^{\infty} \alpha^{\frac{1}{2\ell}} \frac{(-1)^n}{n!} \int_0^{\alpha} t^{n-\frac{1}{2\ell}} dt = \sum \alpha^{\frac{1}{2\ell}} \frac{(-1)^n}{n!} \left. \frac{t^{n+1-\frac{1}{2\ell}}}{n+1-\frac{1}{2\ell}} \right|^{\alpha}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\alpha^{n+1}}{n+1-\frac{1}{2\ell}} \qquad (\text{for } \ell \geq 1) \; .$$

Putting these together, equation (103) becomes

$$A_p = 8\pi m r_+ \left(e^{-\alpha} - \alpha^{\frac{1}{2\ell}} \Gamma(1-\frac{1}{2\ell}) + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\alpha^{n+1}}{n+1-\frac{1}{2\ell}} \right) . \qquad (104)$$

The area in this form (104) can be easily evaluated by computer. A computer program is compiled to this end. The program and the numerical result are listed in Appendix 2, followed by a brief analysis.

V-2) One can also use the method of Euler-Lagrange equation as listed in Chapter 4 to find the minimal surface. This method turns out to be less satisfactory as it involves extremely lengthy and tedious calculation giving an end result that discourages further calculation. However, it does exemplify some finer points. A brief

outline of the calculation is listed below.

The metric of perturbed Kerr 3-geometry is given by

$$ds^2 = e^{2\varepsilon\eta}(r^2+a^2\cos^2\theta)\{\frac{dr^2}{r^2-2mr+a^2} + d\theta^2\} + \ell d\phi^2 \ .$$

As in the Schwarzschild case, we expected that, up to first order of ε, the minimal surface to be the Kerr horizon plus a perturbation term, i.e.

$$r = r_+ + \varepsilon h(\theta) \tag{105}$$

where $h(\theta)$ is to be determined from the Euler-Lagrange equation. The area of a closed 2-surface is given by

$$A = \int e^{\varepsilon\eta}(f^2+a^2\cos^2\theta)\{\frac{f'^2}{f^2-2mf+a^2} + 1\}^{\frac{1}{2}}\ell^{\frac{1}{2}}d\theta d\phi \tag{106}$$

where we have put $r \equiv f(\theta)$. From here, an unsuspecting soul would plunge into calculating the Euler-Lagrange equation until he belatedly discovers, to his horror, that the Lindquist-Boyer coordinates (r,θ,ϕ,t) are unsuitable for such calculation. The metric is singular, a coordinate defect, at $r = r_+$, and hence cannot be used to evaluate the area of the event horizon at $r_+ + \varepsilon h$.

To alleviate this difficulty, we introduce a new coordinate ρ similar to the isotropic coordinate of Schwarzschild geometry by

$$\rho(1 + \frac{r_+}{4\rho})^2 = r \ . \tag{107}$$

In terms of ρ, equation (105) becomes

$$\rho = \frac{r_+}{4} + \varepsilon h(\theta) \tag{108}$$

and the area integral of (106) becomes

$$A = \int e^{\varepsilon\eta}[g^2(1+\frac{r_+}{4g})^4 + a^2\cos^2\theta]\left(\frac{(1+\frac{r_+}{4g})^2 g^2}{g[g(1+\frac{r_+}{4g})^2 - r_-]} + 1\right)^{\frac{1}{2}} \ell^{\frac{1}{2}} d\theta\, d\phi \tag{109}$$

where we have put $\rho \equiv g(\theta)$, and

$$\ell = \left(g^2(1+\frac{r_+}{4g})^4 + a^2 + \frac{2ma^4 g(1+\frac{r_+}{4g})^2\sin^2\theta}{g^2(1+\frac{r_+}{4g})^4 + a^2\cos^2\theta}\right)\sin^2\theta \; .$$

The complexity of this equation deters a normal person from tampering with it. However, if one is brave enough, one can proceed to equate the integrand of (109) to the Lagrangian density L, i.e.

$$L = (\text{integrand of } (109))$$

and demand that L satisfy the Euler-Lagrange equation

$$\frac{\partial L}{\partial g} - \frac{d}{d\theta}\left(\frac{\partial L}{\partial g}\right) = 0 \; . \tag{110}$$

After literally weeks of calculation, this can be shown to reduce to

$$\left(\varepsilon \left.\frac{\partial\eta}{\partial g}\right|_{\rho_+} + \frac{1}{\ell}\left.\frac{d\ell}{dr}\right|_{\rho_+} \cdot \frac{4\varepsilon h}{r_+} + \frac{8\varepsilon h}{(r_+^2+a^2\cos^2\theta)}\right)\left(\frac{r_+}{4}(r_+-r_-) + \varepsilon h(r_+-r_-)\right) - 4\varepsilon h'' = 0 \tag{111}$$

where $\partial\eta/\partial g\big|_{\rho_+}$ means evaluation of $\partial\eta/\partial g$ at $g = \frac{r_+}{4} + \varepsilon h$. Further calculation shows

$$\frac{\partial\eta}{\partial g}\bigg|_{\rho_+} = \frac{\partial\eta}{\partial r}\frac{\partial r}{\partial g}\bigg|_{\rho_+} = \frac{\partial\eta}{\partial r}\bigg|_{\rho_+} \frac{\partial r}{\partial g}\bigg|_{\rho_+}$$

but

$$r = g\left(1 + \frac{r_+}{4g}\right)^2 ,$$

$$\therefore \qquad \frac{\partial r}{\partial g}\bigg|_{\rho_+} = \left(1 + \frac{r_+}{4g}\right)\left(1 - \frac{r_+}{4g}\right)\bigg|_{\rho_+} \simeq \frac{8\varepsilon h}{r_+} .$$

Therefore, in (111) the term $\varepsilon(\partial\eta/\partial g)\big|_{\rho_+}$ is of order ε^2 and can be neglected. Again, as in (105), the equation for the minimal surface does not depend on the perturbation η.

With some more effort, the Euler-Lagrange equation can finally be reduced to

$$h'' - h(r_+)(r_+ - r_-)\left(\frac{4r_+}{r_+^2 + a^2} - \frac{2a^2(r_+ - m)}{(r_+^2 + a^2)^2}\sin^2\theta\right) = 0 . \tag{112}$$

This equation, one notices, has a form similar to the Mathieu equation (Mclachlan, 1964)

$$\frac{d^2y}{dz^2} + (a - 2q\cos^2 z)\,y = 0$$

which can be put in the equivalent form

$$\frac{d^2y}{dz^2} + (b + 4q\sin^2 z)y = 0$$

with $\quad b \equiv a - 2q.$

Even though it is extremely difficult to integrate analytically the area of the minimal surface from equation (112), it is still possible to get a feel of it.

The new coordinate ρ is related to the Lindquist-B oyer coordinate by

$$\rho(1 + \frac{r_+}{4\rho})^2 = r \ .$$

The assertion that ρ changes by order of ε at $r_+/4$ (equation (108)) then implies that r changes by order of ε^2. That is, at $\rho = \frac{r_+}{4} + \varepsilon$,

$$r = (\frac{r_+}{4} + \varepsilon)[1 + \frac{r_+}{4(\frac{r_+}{4} + \varepsilon)}]^2 = r_+ + \frac{4\varepsilon^2}{r_+} + \ldots$$

The area of the minimal surface goes as $\sim r^2$, hence it is a good bet that the area will be independent of terms of order ε. That is the area A will have the form

$$A = A_o + A_1 \varepsilon^2 + \ldots$$

where A_o is the area of the unperturbed Kerr black hole, and A_1 some constant. However, the sign and the magnitude of A_1 can only be determined by direct calculation.

CHAPTER 5

THE GENERALIZED ISRAEL-LEIBOVITZ THEOREM

In this chapter, we generalized the Israel-Leibovitz (1970) theorem to a charged collapsing sphere. This result is then used to show that the Penrose cosmic censorship holds for a charged spherical collapse.

The Israel-Leibovitz theorem (1970) essentially states that assuming an everywhere-positive energy density, a relativistic spherical collapse can never release energy exceeding 100% of its original mass. This limit can only be achieved by bringing each layer of the star momentarily and simultaneously to rest just outside the Schwarzschild radius $r = 2m(r,t)$ corresponding to the mass interior to it. In realistic collapse, the energy released is well below this limit (Israel, 1970).

It should be noted that in Newtonian theory, the energy released through collapse can be arbitrarily large since the gravitational energy is negative and unbounded below.

In the following, the Israel-Leibovitz theorem is generalized to include the case of charged spherical collapse.

The metric outside a charged sphere can be written in curvature coordinates

$$ds^2 = [1 - \frac{2A(r,t)}{r}]^{-1} dr^2 + r^2 d\Omega^2 - [1 - \frac{2A(r,t)}{r}] dt^2 \qquad (1)$$

where $A(r,t) = m(r,t) - \frac{Q^2(r,t)}{2r}$ and can be interpreted as the total mass measured by an external observer at infinity.

These coordinates remain non-singular as long as r remains spacelike and a monotonic function of radial arc length. These conditions are met provided

$$2A(r,t) < r \quad . \qquad (2)$$

We assume that the local energy density is everywhere non-negative, i.e.

$$T_{\mu}{}^{\nu} U_{\nu} U^{\mu} \geq 0 \quad \text{everywhere} .$$

This implies $T_4{}^4 \leq 0$ everywhere. The Einstein field equation then gives

$$G_4{}^4 = -\kappa T_4{}^4 \geq 0 \quad . \qquad (3)$$

In curvature coordinates, $G_4{}^4$ can easily be shown (Synge,1960) to be

$$G_4{}^4 = \frac{2}{r^2} \frac{\partial A(r)}{\partial r} \quad . \qquad (4)$$

Combine this with (3), one gets

$$\frac{\partial A(r)}{\partial r} = -\frac{r^2}{2} \kappa T_4{}^4 \geq 0 \ . \tag{5}$$

In the case where the energy released exceeds 100% of its total mass, the external observer observes negative total mass, i.e.

$$A(R,t) < 0 \tag{6}$$

where R(t) is the boundary of the collapsing star at time t.

In such a case, equation (2) is trivially satisfied. This in turn guarantees that A(r,t) is a monotonic function of r; and hence can be integrated towards the center from equation (5). This gives

$$A(0,t) - A[R(t),t] \leq 0 \ . \tag{7}$$

Writing A(r,t) explicitly,

$$m(0,t) - m[R(t),t] - \left.\frac{Q^2(r,t)}{2r}\right|_{r=0} + \frac{Q^2[R(t),t]}{2R} \leq 0 \ . \tag{8}$$

In the above equation, $Q[R(t),t] \equiv Q$ is the total charge of the sphere; $m[R(t),t]$ is the total bare mass; $Q(r,t)$ is the charge within the radius r. By bare mass, we mean the mass measured when the charge is absent. Since the $Q(r,t) \sim \rho r^3$ where ρ is the charge density,

$$\lim_{r\to 0} \left[\frac{Q^2(r,t)}{2r}\right] \sim \lim_{r\to 0} \left[\frac{r^6}{r}\right] = 0 \ . \tag{9}$$

Equation (6) implies $m(R,t) < 0$. This and (9), combined with (8) imply

$$m(0,t) < m[R(t),t] < 0 \quad . \tag{10}$$

This result contradicts with our earlier assumption that the local energy density is everywhere non-negative since $m(0,t) < 0$ necessarily implies a negative energy density in the neighbourhood of $r = 0$. In other words, A[R(t),t] can never be negative, and the efficiency of energy release is always less than 100%.

A charged spherical black hole must necessarily have $Q < M$. If in the process of collapse, a charged sphere with a charge Q smaller than the initial mass M_i were able to lose enough mass so that its charge exceeds its mass, a naked singularity would be formed. The above theorem, however, says this can never happen.

In the process of collapse, the condition $A[R(t),t] \geq 0$ must hold. Or writing $A(r,t)$ explicitly,

$$m(R,t) - \frac{Q^2(R,t)}{2R} \geq 0$$

or

$$R \geq \frac{Q^2(R,t)}{2m(R,t)} \quad . \tag{11}$$

Since $Q(R,t)$ is the total charge, $m(R,t)$ is the bare mass at time t, equation (11) implies that the collapse can never go beyond $R = Q^2/2m$. A singularity is thus avoided.

This, of course, does not mean that an event horizon will not be formed. In fact, the minimum radius R of (11) is strictly less than the inner event horizon of a Reissner-Nordstrom black hole.

APPENDIX 1

In equation (4.31), it is shown that the metric at $t = 0$ for an axisymmetric (t,ϕ) reversible spacetime is given by

$$ds^2 = e^{2\psi}(d\rho^2+dz^2) + \ell d\phi^2 + 2md\phi dt - fdt^2 \ . \tag{4.31}$$

To further facilitate calculation, three new functions, $\tilde{\nu}$, $\tilde{\omega}$, σ can be introduced to reduce (4.31) to (4.35). The new functions are defined by

$$\tilde{\nu} \equiv \psi + \tfrac{1}{2} \ln \ell \tag{4.32}$$

$$\tilde{\omega} \equiv m/\ell \tag{4.33}$$

$$(\sigma\rho)^2 \equiv -\det\begin{vmatrix} g_{33} & g_{43} \\ g_{43} & g_{44} \end{vmatrix} = f\ell + m^2 \ . \tag{4.34}$$

In the following, we show that, theoretically at least, there exists a coordinate transformation which reduces σ to 1, thus reducing the number of unknown functions from four to three.

The (t,ϕ) reversibility is not effected by a transformation

$$t = t'F(x^A) \ , \tag{A1}$$

or equivalently,

$$dt = dt'F(x^A) + t'dF(x^A) \ . \tag{A2}$$

Or the initial hypersurface $\sum_i$: $t = 0$, the transformation

is

$$dt = dt'F(x^A) \quad . \tag{A3}$$

Substitute into (4.31) gives

$$ds^2 = e^{2\psi}(d\rho^2+dz^2) + \ell d\phi^2 + 2mFd\phi dt' - fF^2(dt')^2 \quad . \tag{A4}$$

Introduce new functions $\tilde{\nu}$, $\tilde{\omega}'$, σ' by

$$\tilde{\nu} \equiv \psi + \tfrac{1}{2}\ln \ell \tag{A5}$$

$$\tilde{\omega}' \equiv mF/\ell \tag{A6}$$

$$(\sigma'\rho)^2 \equiv -\det\begin{vmatrix} g_{33} & g_{43} \\ g_{43} & g_{44} \end{vmatrix} = F^2(f+m^2) \quad . \tag{A7}$$

Compare (A4), (A6),(A7) with (4.31), (4.33), (4.34). One finds the correspondence between various functions:

$$\tilde{\omega}' \rightarrow \tilde{\omega}F \tag{A8}$$

$$\sigma' \rightarrow \sigma F \tag{A9}$$

$$f \rightarrow fF^2 \quad . \tag{A10}$$

By artfully choosing F, σ' can be reduced to 1.

With this particular choice of F, the metric of (A4) can be written as

$$ds^2 = \ell^{-1}e^{2\nu}[d\rho^2+dz^2) + \ell d\phi^2 + 2\ell\phi' d\, dt' - \ell^{-1}[\rho^2-\ell^2(\tilde{\omega}')^2] \times (dt')^2 \quad . \tag{A11}$$

In this form, the metric is identical with the metric of the stationary, axisymmetric, (t,ϕ) reversible case.

APPENDIX 2

In this appendix, we list the numerical computation of equation (4.104),

$$A_p = 8\pi m r_+ \{e^{-\alpha} - \alpha^{\frac{1}{2\ell}} \Gamma(1 - \frac{1}{2\ell}) + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\ell^{n+1}}{n+1-\frac{1}{2\ell}}\}$$

where A_p is the area of the minimal surface in a slightly perturbed Kerr 3-geometry.

In the computer output, $A \equiv \alpha$, $L \equiv \ell$ and the unperturbed minimal surface, i.e. $\alpha = 0$, has an area of 2.

Once again, the area of the perturbed minimal surface is strictly less than the unperturbed area (compare with Schwarzschild case in Chapter 3). This again confirms Penrose's cosmic-censor conjecture.

A= 0.00000

L=1	2.000000
L=2	2.000000
L=3	2.000000

A=	0.10000D-04	0.20000D-04	0.30000D-04	0.40000D-04	0.50000D-04
L=0	2.000000	2.000000	2.000000	2.000000	2.000000

A=	0.10000D-05	0.20000D-05	0.30000D-05	0.40000D-05	0.50000D-05
L=1	1.996457	1.994991	1.993866	1.992918	1.992083
L=2	1.922499	1.907835	1.898003	1.890398	1.884111
L=3	1.774243	1.746597	1.728881	1.715565	1.704788
L=4	1.612459	1.577384	1.555412	1.539134	1.526099
L=5	1.463145	1.424614	1.400805	1.383317	1.369401

A=	0.10000D-03	0.20000D-03	0.30000D-03	0.40000D-03	0.50000D-03
L=1	1.964751	1.950267	1.939200	1.929902	1.921733
L=2	1.754983	1.708678	1.677652	1.653666	1.633848
L=3	1.513660	1.454137	1.416009	1.387360	1.364181
L=4	1.310872	1.248526	1.209483	1.180564	1.157411
L=5	1.149165	1.088118	1.050405	1.022710	1.000678

A=	0.10000D-02	0.11000D-02	0.12000D-02	0.13000D-02	0.14000D-02
L=1	1.889900	1.884628	1.879600	1.874786	1.870161
L=2	1.564840	1.554397	1.544648	1.535495	1.526859
L=3	1.286492	1.275101	1.264546	1.254702	1.245473
L=4	1.081280	1.070295	1.060153	1.050727	1.041918
L=5	0.929056	0.918820	0.909392	0.900647	0.892490

A=	1.0000	1.1000	1.2000	1.3000	1.4000
L=1	0.178148	0.152628	0.131213	0.113140	0.097813
L=2	0.098493	0.083968	0.071870	0.061726	0.053173
L=3	0.068006	0.057873	0.049456	0.042416	0.036492
L=4	0.051922	0.044146	0.037694	0.032305	0.027775
L=5	0.041989	0.035680	0.030451	0.026085	0.022419

A= 0.00000

L=1	2.000000
L=2	2.000000
L=3	2.000000

A=	0.60000D-04	0.70000D-04	0.80000D-04	0.90000D-04
L=0	2.000000	2.000000	2.000000	2.000000

A=	0.60000D-05	0.70000D-05	0.80000D-05	0.90000D-05
L=1	1.991329	1.990635	1.989989	1.989383
L=2	1.878707	1.873941	1.869663	1.865768
L=3	1.695680	1.687761	1.680734	1.674405
L=4	1.515175	1.505742	1.497423	1.489970
L=5	1.357799	1.347823	1.339056	1.331225

A=	0.60000D-03	0.70000D-03	0.80000D-03	0.90000D-03
L=1	1.914368	1.907610	1.901335	1.895452
L=2	1.616824	1.601820	1.588354	1.576103
L=3	1.344597	1.327574	1.312475	1.298879
L=4	1.138013	1.121268	1.106503	1.093276
L=5	0.982311	0.966522	0.952649	0.940260

A=	0.15000D-02	0.16000D-02	0.17000D-02	0.18000D-02
L=1	1.865706	1.861403	1.857239	1.853201
L=2	1.518678	1.510900	1.503481	1.496385
L=3	1.236781	1.228560	1.220760	1.213334
L=4	1.033645	1.025842	1.018457	1.011444
L=5	0.884843	0.877642	0.870837	0.864385

A=	1.5000	1.6000	1.7000	1.8000
L=1	0.084758	0.073600	0.064030	0.055799
L=2	0.045927	0.039762	0.034497	0.029987
L=3	0.031482	0.027227	0.023599	0.020495
L=4	0.023948	0.020700	0.017932	0.015567
L=5	0.019322	0.016696	0.014460	0.012549

BIBLIOGRAPHY

General Texts

Classical Theory of Fields, 1971, revised 2nd English edition, Landau, L.D. and Lifshitz, E.M., Addison-Wesley, Massachusetts, U.S.A.

Gravitation, 1973, Misner, C.W., Thorne, K.S. and Wheeler, J.A., Freeman and Company, California, U.S.A.

Gravitation and Cosmology: Principles and Applications of The General Theory of Relativity, 1972, Weinberg, S., Wiley and Sons, Inc., New York, U.S.A.

Introduction to General Relativity, 1965, Adler, R., Bazin, M. and Schiffer, M., McGraw-Hill, New York, U.S.A.

Large Scale Structure of Spacetime, 1972, Hawking, S.W. and Ellis, G.F.R., Cambridge University Press, Cambridge, England.

Relativity; The General Theory, 1960, Synge, J.L., North-Holland, Amsterdam.

Riemannian Geometry, 1966, Eisenhart, L.P., Princeton University Press, Princeton, U.S.A.

Articles

Arnowitt, R., Deser, S. and Misner, C.W. (1962) "The Dynamics of General Relativity", in Gravitation: An Introduction to Current Research (ed. by L. Witten), Wiley, New York, U.S.A.

Brill, D.R. (1959) Annals of Physics, 7, 466-483.

Cameron, A.G.W. (1970) Ann. Rev. Astron. & Astrophys., 8, 179-208.

Eisenhart, L.P. (1966) "Riemannian Geometry", Princeton University Press, Princeton, U.S.A.

Ernst, F.J. (1968), Phys. Rev.,5, 167, 1175-1178.

Gibbons, G.W. (1972) Commun. Math. Phys., 27, 87-102.

Harrison, B.K., et. al. (1965) "Gravitation Theory and Gravitational Collapse", University of Chicago Press, Chicago, U.S.A.

Hartle, J.B. (1972) "Relativity, Astrophysics and Cosmology" (ed. by W. Israel), D. Reidel, Holland.

Hawking, S.W. (1967) Proc. Roy. Soc. A300, 187.

Hawking, S.W. and Ellis, G.F.R. (1972) "Large Scale Structure of Spacetime", Cambridge University Press, Cambridge, England.

Hawking, S.W. (1972a) Commun. Math. Phys., 25, 152-166.

Hawking, S.W. (1973) Commun. Math. Phys., 33, 323-334.

Israel, W. (1966) Nuovo Cimento, 44B, 1-14.

Israel, W. and Leibovitz, C. (1970) Phys. Rev. D 12, 1, 3226-3228.

Israel, W. (1975) personal communication.

Khan, K.A. (1964), M. Sc. Thesis, University of Alberta, Edmonton.

Kruskal, M.D. (1960) Phys. Rev., 119, 1743-1745.

Kuchar, K. (1972) "Relativity, Astrophysics and Cosmology" (ed. by W. Israel), D. Reidel, Holland.

Oppenheimer, J.R., et.al. (1939) Phys. Rev., 55, 374.

Penrose, R. (1965) Phys. Rev. Lett., 14, 57-59.

Penrose, R. (1969) Rivista Del Nuovo Cimento, 1, 252-276.

Penrose, R. and Hawking, S.W. (1969) Proc. Roy. Soc. London, A314, 529-548.

Penrose, R. (1971), Seminar given at Cambridge University.

Price, R.H. (1972) Phys. Rev. D 5, 2419-2454.

Rhodes, C.E.J. (1971) Ph. D. dissertation, Princeton University.

Rhodes, C.E.J. and Ruffini, R. (1972) "On the maximum mass of a neutron star" in "Black Holes" (ed. by DeWitt and Dewitt), Gordon and Breach.

Seifert, H.J., et. al. (1973) Commun. Math. Phys., 34, 135-148.

Seifert, H.J., et. al. (1974) Commun. Math. Phys., 37, 29-40.

Szekeres, G. (1960), Publ. Mat. Debrecen, 7, 285-301.

Weber, J. and Wheeler, J.A. (1957) Rev. Mod. Phys., 29, 509.

Weyl, H. (1917) Ann. Physik, 54, 117.